Structuralism and the Indispensability Argument

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Abstract:

I distill the indispensability argument to four general charateristics, which yield six unfortunate consequences, unwelcome characteristics of the objects yielded by the argument. The indispensabilist's objects are not mathematical, in any traditional sense, but merely empirical posits. Then, I characterize a form of structuralism in the philosophy of mathematics which relies on an indispensability argument. I show that its yield suffers the unfortunate consequences.

I consider a rejoinder from Michael Resnik, that the traditional objects of mathematics are also empirical. I show in detail that Resnik provides no reason to blur the line between abstract, mathematical objects, and concrete physical ones. Lastly, I present some further considerations opposing the structuralism to which Resnik subscribes. Resnik's structuralism, and its indispensability argument, should be avoided.

§1: Introduction

Quine's indispensability argument alleges that the construction of scientific theory forces us to commit to the existence of mathematical objects. While it is obvious that scientists use mathematics in developing their theories, it is far from obvious why the uses of mathematics in science should force us to believe in the existence of abstract objects. If we want to study the interactions of charged particles, say, we might rely on Coulomb's Law, which refers to a real number, and employs mathematical functions including multiplication and absolute value. Still, we use Coulomb's Law to study charged particles, not to study mathematical objects, which have no effect on those particles.

The plausibility of Quine's indispensability argument thus relies on his claim that our ontic commitments are just those of the first-order theory which we can construct to embody our best scientific theory. We gather our physical laws and cast them in first-order logic with identity. The commitments of this regimented theory may be found by examining its quantifications.[1] I believe that Quine's method for determining ontic commitment does not accurately yield our commitments, but I shall not argue that here.

Other versions of the indispensability argument are available, some of which eschew Quine's method for determining ontic commitment. In this paper, I show that some casual appeals to the uses of mathematics in science, ones which do not depend on Quine's method, do not justify mathematical realism. In specific, I show that a version of structuralism developed by Michael Resnik and Stewart Shapiro both relies on an indispensability argument, and suffers

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[1] In particular, the commitments of a theory may be found by examining its domain of quantification. The domain of quantification is technically an element of the metatheory, not the theory itself.

from problems which beset all indispensability arguments.

To show that the structuralist is an indispensabilist, I first distill the indispensability argument to its essential characteristics, and note the unfortunate consequences that result.

§2: The Essential Characteristics of Indispensability Arguments

The following are Essential Characteristics of any indispensability argument for mathematical realism.[2]

EC.1: Naturalism: The job of the philosopher, as of the scientist, is exclusively to understand our sensible experience of the physical world.
EC.2: Theory Construction: In order to explain our sensible experience we construct a theory of the physical world. We find our commitments exclusively in our best theory.
EC.3: Mathematization: We are committed to some mathematical objects and/or the truth of some mathematical statements, since they are essential to that best theory.
EC.4: Subordination of Practice: Mathematical practice depends for its legitimacy on empirical scientific practice.

It follows from EC.1 that we never need to explain mathematical phenomena, like the existence of unexpectedly many twin primes, for their own sake. Ultimately, the justification for any mathematical knowledge must appeal to an account of our sense experience.

EC.2 indicates a general source of ontic commitment, but does not determine a particular criterion. If we drop Quine's method for determining ontic commitment, we need another way to determine the commitments of our best theory. EC.2 rules out independent appeal to an

[2] I develop these more fully elsewhere. Here, take them as defining the indispensabilist's position.

autonomous mathematical theory for justification of mathematical claims.[3]

EC.3 is an empirical claim about the needs of theory construction. I call this claim "empirical" since it is an open, and it seems to me empirical, question whether we can formulate nominalist alternatives to all future scientific theories. The indispensabilist relies on speculation that this is impossible.

EC.4 is actually implicit in the other characteristics, but it emphasizes the relationship between mathematics and empirical science for the indispensabilist. Dropping EC.4 from an indispensability argument would entail either adopting an alternate justification for mathematical commitment and practice, or denying that mathematical practice yields any commitments.

§3: The Unfortunate Consequences, and their Links to the Essential Characteristics

The Essential Characteristics entail some Unfortunate Consequences for the mathematical realism generated by any indispensability argument. First, since Mathematization and Naturalism rule out any alternate justifications for mathematical claims, the indispensabilist has no commitments to mathematical objects which are not required for science.[4] I call this Unfortunate Consequence 'Restriction.' It is difficult to say precisely which mathematical objects the indispensability argument would justify, i.e. how much mathematics science actually

[3] One might think that an indispensabilist may also admit an autonomous epistemology for mathematics. He would seem to have two routes to mathematical knowledge, one independent of empirical science, and the other relying on it. But, the former route would do all the real work. Science might explain our beliefs, but it would not justify our knowledge.

[4] Supplementing the indispensability argument to justify unapplied results, e.g. by appeal to a priori intuition or logicism, renders the original argument superfluous. See fn 3.

needs. Burgess and Rosen offer the following: “It has been the received view and expert opinion among competent logicians since the 1920s that the mathematics needed for applications can be developed in a theory known as mathematical analysis, in which the only entities mentioned are real numbers.” (Burgess and Rosen (1997) p 76)

The point at which the indispensabilist draws the line is unimportant. What is relevant is the existence of a gap, one which Quine countenances. “I recognize indenumerable infinities only because they are forced on me by the simplest known systematizations of more welcome matters. Magnitudes in excess of such demands, e.g., $\beth_\omega$ or inaccessible numbers, I look upon only as mathematical recreation and without ontological rights.” (Quine (1986) p 400)

The restrictions on the indispensabilist do not merely apply to the outer regions of set theory. Justifications of mathematical claims vary with shifts in our best scientific theory. As science progresses, and uses new mathematical tools, the mathematics which is justified changes, though no mathematical progress need be made.

The adoption of mathematics with no further mathematical justification is not the worst problem arising from the restrictions. We could see a profound upheaval of the justifications of mathematics. Maddy suggests that all of science could, in principle, become quantized.[5] We would lose continuum mathematics, the calculus and analysis. Not only is the line between justified and unjustified results drawn in the wrong place, but it could move.

Another Unfortunate Consequence arises directly from Theory Construction, which entails that the indispensabilist can not successfully differentiate between abstract and concrete objects. He may call some of his commitments abstract and others concrete, but these are empty

[5] Maddy (1992), p 285-6.

labels, for the indispensabilist. All commitments are made in the same way, for the same purpose, to account for sensory experience. We should classify the indispensabilist's purported abstract objects with the concrete objects they are used to explain or describe. I call this Unfortunate Consequence 'Ontic Blur.'

A mathematical realist who does not rely on the indispensability argument can establish a criterion for abstractness by distinguishing the disciplines of mathematics and empirical science. Since the epistemology for mathematics is separate from that of empirical science, and the ontologies are distinct, the claim that mathematical objects are abstract is plausible.[6]

When we combine Theory Construction with Mathematization, we find that the indispensabilist's mathematical objects do not exist necessarily. This 'Modal Uniformity' is ironic, since the indispensabilist's claim that mathematics can not be excised from science includes an appeal to modality.

There are several notions of necessity. A statement may be metaphysically necessary, or true in all possible worlds. A statement may be logically necessary, which may be construed as true by definition, or, more formally, as entailing a contradiction when negated. These are stronger notions than physical necessity, or following from the laws of physics. When one asserts that the world is possibly Newtonian, even if relativistic, one relies on the weaker physical necessity, on which phenomena in accord with scientific laws follow from them necessarily. By linking mathematics to the physical world, the indispensabilist may retain this weaker notion. Unfortunately, the weaker notion is not the one traditionally imputed to mathematics, and is unsatisfactory. It would follow that under a different set of physical laws,

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[6] See §5.3 for more on the abstract/concrete distinction.

two and two might not be four. While this idea may be alluring to some, it seems absurd. A stronger necessity is to be preferred, if only to appease our intuition that mathematical truths are broader than physical ones. Modalities are notoriously intractable. Certainly any Quinean will be wary of any modality, besides logical necessity. But mathematical realism with only a weak notion of necessity conflicts with our commonsense beliefs about mathematics.

Mathematical objects become temporal, as well as contingent, as a corollary of Modal Uniformity, since the contingency of mathematical objects entails that there can be a time when they do not exist. Like contingency, the temporality of mathematical objects is counter-intuitive. “It would betray a confusion to ask, ‘When did (or when will) these primes exist? At what time may they be found?” (Burgess and Rosen (1997) p 21)

The combination of Theory Construction and Mathematization also entails that mathematical objects are known a posteriori. While it is traditional to ascribe to mathematics an a priori methodology, the indispensabilist only provides an epistemology for empirical science.

Lastly, Subordination of Practice means that any mathematical debate, like that over the Axiom of Choice, should be resolved not on mathematical terms, but on the basis of the needs of science. In Resnik’s terms, the indispensabilist’s appeal to “Euclidean rescues” is limited. When relativity supplanted classical mechanics, Euclidean geometry became inapplicable to physical space, and was supplanted. A Euclidean rescue defends both the new mathematical theory and the old one despite the change in physical theory. All three possibilities concerning the parallel postulate are taken as consistent, and unfalsified, theories.[7]

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[7] Euclid’s parallel postulate states that if a line intersects two other lines and makes the interior angles on the same side less than two right angles, then the two lines meet on that side. The parallel postulate is equivalent to Playfair’s Postulate, which states that given a line and a

We can perform a Euclidean rescue any time a mathematical theory fails to apply in science. In such cases, the indispensabilist generally rejects the now-unapplied mathematics. The traditional realist generally chooses the Euclidean rescue, unless the mathematics is shown inconsistent.

Consider two conflicting mathematical theories, like ZF + CH and ZF + not-CH.[8] It is possible, in this cases and others like it, that each of the conflicting pairs will find some physical application.

The indispensabilist commits to a mathematical theory only if it has a physical application. This application need not be to any fundamental situation; any application will do. If only one of the conflicting pairs applies, there is no problem in deciding which theory to accept. In the hypothetical case we are considering, the indispensabilist is committed to both theories, and must find some consistent interpretation. The traditional realist multiplies universes. Perhaps there are multiple set-theoretic hierarchies; in some the continuum hypothesis holds while in others it fails, and in multiple different ways. The indispensabilist is committed to austerity in abstracta and accommodates applications of conflicting mathematical theories more awkwardly.

As a last illustration of UC.5, consider the introduction of complex numbers, as solutions to quadratic equations with missing real roots. 'Imaginary' or 'impossible' numbers were

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point not on that line, exactly one line can be drawn through the given point parallel to the given line. There are two ways to deny Playfair's postulate, or the parallel postulate, both of which are consistent with the other axioms of geometry. If one can draw no parallel lines, the geometry defines the surface of a sphere. If one can draw more than one parallel line, one defines a surface called a hyperbolic spheroid, or a pseudo-sphere.

[8] That is, Zermelo-Frankel set theory with the continuum hypothesis, or with its negation.

derided, despite their mathematical virtues. They simplified mathematics, since ad hoc explanations about why certain quadratic equations had two roots, others just one, and others none, were avoided. A fruitful field of study was born with geometric, graphical representations. The theory of complex numbers was not found to contain any inconsistency, aside from the conflict with a presupposition that all numbers were real numbers. Physical applications were later discovered, for example in representing inductance and capacitance as the real and imaginary parts of one complex number, instead of as two distinct reals.

For the mathematician, and the traditional realist, the legitimacy of complex numbers came early. The indispensabilist, prior to the discovery of their applicability, could make no room for them. Even the analogy with negative numbers, which arose from similar disgrace, serves as no argument for the indispensabilist. Lacking application, work with complex numbers was just mathematical recreation.

We can be sure that mathematicians working today in the farthest reaches of pure mathematics do so without knowing that their work has any physical application. One may arise, or their work may lie fallow. If the only justification for mathematics is in its application to scientific theory, then unapplied results are unjustified, even if they may eventually be useful. The indispensabilist makes the mathematician dependent on the scientist for the justification of his work.

Chihara criticizes the subordination of mathematical practice which results from Quine's argument. "It is suggested [by Quine] that which mathematical theory we should take to be true should be determined empirically by assessing the relative scientific benefits that would accrue to science from incorporating the mathematical theories in question into scientific theory. It is as if the mathematician should ask the physicist which set theory is the true one!" (Chihara (1990) p 15)

The following are thus the Unfortunate Consequences of any mathematical realism generated by an indispensability argument:

UC.1: Restriction: Our commitments are to only those mathematical objects required by empirical science. Mathematical results which are not applied in scientific theory are illegitimate. Mathematical objects not required for applied mathematics do not exist.
UC.2: Ontic Blur: Mathematical objects are concrete.
UC.3: Modal Uniformity: Mathematical objects do not exist necessarily.
UC.4: Temporality: Mathematical objects exist in time.
UC.5: Aposteriority: Mathematical objects are known a posteriori.
UC.6: Uniqueness: Any debate over the existence of a mathematical object will be resolved by the unique answer generated by empirical theory.

§4: Structuralism as Indispensabilism

In this section, I characterize a form of structuralism, and show how it relies on an indispensability argument. I use 'structuralism' to refer to a limited number of positions which might be called structuralist. I use it to refer to a realist philosophy of mathematics which focuses on the existence of structures, or patterns, defined by mathematical theorems.[9] Some mathematical realists are "object realists," since they focus on the existence of mathematical objects. The structuralist is a "sentence realist," focusing on the truth of axioms and theorems of mathematics. The structuralist can either eschew commitment to mathematical objects, or derive it from commitments to structures.

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[9] Anti-realist versions of structuralism are possible, as well as ontically neutral ones. See Hellman (1989). These are not my concern, here.

The structuralist adopts his position to gain advantage in answering the main epistemic problem facing mathematical realists. The epistemic problem for realists is to describe how we can have knowledge of abstract objects or of truths which refer to abstract objects. The indispensabilist solves this problem by referring all justification to empirical theory. The structuralist adopts an indispensability argument when he refers to empirical evidence to justify knowledge of structures.

Structuralism, as a philosophy of mathematics, became popular in the wake of Benacerraf (1965) as an attempt to dissolve the problem of choosing which sets are the numbers. Benacerraf's problem was that different sets of sets can model the Peano axioms, and nothing can tell us which one of these sets is the right reduction. It seemed, to Benacerraf and others, that object realism commits us to indeterminacy. Since the structuralist is committed only to the truths of the axioms and theorems, he need not commit to any particular models, and so the problem of indeterminacy is dissolved.

There are other ways around Benacerraf's problem. A sui generis solution avoids the problem of multiple set-theoretic models of the Peano axioms by denying that we should expect any set-theoretic model to be uniquely correct. Number-theoretic axioms are modeled by the numbers, and there are various set-theoretic dopplegangers.

In an attempt to avoid the Benacerraf problem, the structuralist substitutes a structure, defined by the axioms, in lieu of individual objects. But, a structure is merely a more complex abstract object, just like the set of numbers. Furthermore, within structures, there are nodes, or points. Within each member of the class of structures defined by the Peano Axioms, for example, are positions which correspond to each of the natural numbers. If we have knowledge of the structure, then we would seem to have knowledge of the positions in the structure, and thus have knowledge of individual mathematical objects after all.

The epistemic problem facing realism arises whether the objects to which the realist is committed are numbers or structures. If the structuralist commits only to concrete structures, he has not generated mathematical realism. If he purports to generate mathematical realism, he must account for our access to abstract structures. Shifting the focus of the debate does not remove the challenge for the realist.

So, the epistemic problem facing the structuralist is how to account for our knowledge of structures. Resnik tries to solve the problem, in part, by denying that structures are objects.[10] He provides no criteria for identity among structures, for example. Resnik argues that we should expect that structures are not objects in the same way that we expect numbers not to be sets. There is no fact of the matter about whether two structures are identical, or whether a position in one structure is identical to a position in another. He maintains, though, that positions in patterns are mathematical objects, like numbers, and that these exist.

Denying that structures are objects comes at the cost of any explanation of how we know about mathematical objects. For, if we can not know about structures as objects, then we need a separate account of how we know about their positions. Resnik can not make any headway in answering the epistemic question facing realism by denying that structures are objects.

Resnik also provides an historical account of our experience with concrete structures to account for our mathematical beliefs. He alleges that we have a basic understanding of concrete

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[10] Denying that structures are objects also helps Resnik avoid a problem of pattern self-inclusion. See Resnik (1997) p 256.

patterns, like a template for drawing or a chessboard. We generalize, or idealize, to the notion of an abstract, underlying structure. Our ideas about structures and their relations apply both to concrete and abstract instances. "*Pattern congruence* is an equivalence relation whose field I take to include both abstract mathematical structures and arrangements of more concrete objects." (Resnik (1997) p 204) The abstract structure forms the basis of classical mathematics, but we need not posit any access to this structure beyond our physical, causal connection to token patterns.

Maddy, finding the indispensabilist account unsatisfactory, proposed actual perception of abstracta.[11] Resnik avoids committing to an implausible Maddy-esque perception. "[W]e gain 'access' to mathematical objects by positing them and correlations between some of their features and concrete computations... [T]he features in question are structural." (Resnik (1997) p 87) While Resnik's postulational account is inessential to indispensability, it reveals the structuralist's connection between mathematics and its applications. The structuralist portrays the introduction of abstracta as an implicit or explicit posit, on the road from actual to possible concreta.

Resnik justifies the leap to abstract objects on the basis of fruitfulness. "I hypothesize that using concretely written diagrams to represent and design patterned objects, such as temples, bounded fields, and carts, eventually led our mathematical ancestors to posit geometric objects as sui generis. With this giant step behind them it was and has been relatively easy for subsequent mathematicians to enlarge and enrich the structures they knew, and to postulate entirely new ones." (Resnik (1997) p 5)

[11] See Maddy (1990).

Resnik's quasi-historical account of mathematical beliefs does not justify them. It shows how the structuralist attempts to erode the abstract/concrete distinction to establish ontic blur. If the line between abstract and concrete objects is eroded, then an account of those elements of our ontology ordinarily deemed abstract may be continuous with our account of those we call concrete.

Resnik admits that there is a difference between mathematical and scientific positing, and notes that scientists posit to explain previously observed phenomena, and devise experiments to detect new posits. In contrast, we have a local conception of mathematical evidence, which relies on proof, computation, and logic. "In practice, when justifying a mathematical claim, we hardly ever invoke such global considerations as the benefits to natural science. We ordinarily argue for pieces of mathematics locally by appealing to purely mathematical considerations." (Resnik (1997) p 6)

But, Resnik argues, observation is also relevant to mathematics.

> It would be wrong to conclude from its possessing a local conception of evidence that mathematics is an apriori science, disconnected evidentially from both natural science and observation. ...[W]hen supplemented with auxiliary hypotheses, mathematical claims yield results about concretely instantiated structures, such as computers, paper and pencil computations, or drawn geometric figures, that can be tested observationally in the same way that we test other scientific claims. (Resnik (1997) p 6)

For Resnik, even simple acts like counting become confirmation for mathematical theories. "Practice with counting, measuring, surveying, and carpentry suggested and confirmed the elementary rules of practical arithmetic and geometry long before they were elevated to the status of inviolable laws and codified into mathematical systems." (Resnik (1997) p 48)

Resnik alleges that we test mathematical theories as directly as we test scientific theories.

We reduce their empirical content. We can construct Turing machines. While we can not test all mathematical theories empirically, we can not test all scientific ones, either, like the hypothesis that space is continuous.

Shapiro (1983b, 1997) presents structuralism as primarily an account of the application of mathematics to physical theory. He argues that the traditional philosophies of mathematics (i.e. formalism, logicism, traditional platonism, and intuitionism) provide either no explanation of the relationship between mathematics and physical reality or an unsatisfactory one. According to Shapiro, structuralism solves the problem, in part by providing a more holistic view of the relationship between of mathematics and science. We have experience with concrete instantiations of structures, and this provides insight into the abstract natures of structures, the subject matter of mathematics. "My view is that, extensionally speaking, there is no difference, or at any rate no philosophically illuminating difference [between mathematical structures and other kinds of structures]." (Shapiro (1983b) p 542)

If there is no difference between these two types of structures, we can infer knowledge of mathematical structures on the basis of our knowledge of physically instantiated structures. The inference to mathematical knowledge from knowledge of the physical world is the fundamental claim of the indispensability thesis.

Both Shapiro and Resnik move from the exemplification of mathematical structure in the physical world, which is studied by physical science, to our knowledge of abstract mathematics. The structuralist says that abstract mathematics applies to physical science because mathematical structures are exemplified in the physical world. Since this is intended as an account of the applicability of mathematics to science, it is also an account of our knowledge of mathematics.

Structuralism posits no epistemology for mathematics other than that which we already need for our knowledge of the empirical world. It purports to generate mathematical realism. It satisfies all the Essential Characteristics EC.1 - EC.4, though it may restrict EC.3, Mathematization, only to truth for sentences.

Given that structuralism has all the Essential Characteristics, its mathematical realism is saddled with all the Unfortunate Consequences. Though the structuralist avoids Uniqueness for some questions, like the identification of numbers with sets, he can not avoid committing to a fact of the matter about others, like that of the status of non-Euclidean geometries. For any indispensabilist, only the geometry used in our best physical science is justified.

Attempting to make lemonade out of lemons, the indispensabilist embraces some of the Unfortunate Consequences. In the next section, I show how Resnik tries to argue for ontic blur. He also denies that the structuralist's limited ontology, its Restriction, is a problem. "[Science] falls short of affirming the existence of many of the entities studied by the far reaches of contemporary mathematics. I do not find this a drawback at this point..." (Resnik (1993) p 58)

It is a drawback, and Resnik later recognizes it. "Axioms limiting the size of the set-theoretic universe would discourage the development of mathematics through limiting the structures it recognizes. Furthermore, while limiting the variety of structures would probably not hinder contemporary science, it might hinder future science. So the good of neither mathematics nor science as a whole calls for adding limitative axioms to set theory." (Resnik (1997) p 147)

An indispensabilist can claim only those objects necessary to scientific theory, in this case to our understanding of concrete structures. Resnik argues that restrictions might limit future science. If future science demands an extension, the restriction in this case will be eased.

For now, we do not know which results might be needed. Despite his efforts to avoid it, Resnik's structuralist suffers from Restriction.

§5: Structuralism's Supplement: Ontic Blur

At first glance, the structuralist's account of mathematics based on sensory experience of concrete patterns seems as unlikely to be fruitful as the approaches which start with apprehension of discrete physical objects, or inscriptions of rough shapes. All of these approaches stumble on the leap from knowledge of concreta to knowledge of abstracta.

Resnik adds an argument for ontic blur to bridge the gap. If concrete and abstract objects are no different in kind, then there is no leap to be made. Our knowledge of abstract objects would be like that of concrete objects. If the structuralist can establish ontic blur, concomitantly no objects exist necessarily, and an argument for UC.3, Modal Uniformity, is also established.[12]

Central to Resnik's argument for blur is his connection between physical explanation and mathematical objects. "Mathematical facts and properties of mathematical objects play essential roles in physical explanations themselves." (Resnik (1993) pp 42-3) Consider a ball thrown straight up in the air. The explanation of why it stops and turns around at a particular point, when the vector sum of the upward and downward velocities equals zero, relies on mathematical properties of the ball's velocity. "Moreover, this velocity itself, being a function, is a mathematical object. So, the explanation uses mathematical objects and their properties to explain the behavior of a physical thing." (ibid, p 43)[13]

[12] Similar arguments hold for UC.4, Temporality, and UC.5, Aposteriority.

[13] Shapiro makes the same argument in Shapiro (1983b).

Resnik uses the above example to argue that mathematical objects not only play a role in a physical explanation, but they have efficacy in the spatio-temporal world. Physical objects and mathematical ones are linked in the world.

The question of whether mathematical objects can participate in physical events raises the question of what the mathematical objects are. We can agree on some core objects: numbers, sets, and geometric points. Resnik extends the list, though his claims conflict with our ordinary conception of mathematical objects. He claims that velocity, quantum particles, and fields qualify as mathematical. "It can be unclear whether a given explanatory object (for example, a field) is physical or mathematical, or even whether something counts as physical behavior (for example, the collapse of a field)." (Resnik (1993) p 45)

I examine Resnik's arguments that traditional attempts to distinguish between mathematical and physical objects are unsatisfactory. He argues that mathematical objects, like physical objects, may change their properties, participate in events, and be located in space and time. I show how the mathematical objects are in fact unlike physical objects, and how physical objects are unlike mathematical ones, thus refuting the purported blur.

§5.1: Mathematical Objects and Property Change

In addition to existing necessarily, mathematical objects are traditionally distinguished from physical objects by having their properties necessarily. Resnik argues for blur by arguing that mathematical objects, like physical ones, can change properties. While some properties of mathematical objects may change, this does not block the distinction between mathematical and physical objects.

Resnik presents an example of how a number can change its properties. "Just as Smith may be thin as a child, and not as an adult, the number 60 may register Smith's height in inches at age 12 and not at a later age." (Resnik (1997) p 108)

Even if we accept such Cambridge properties as real, we can still distinguish between mathematical and physical objects on the basis of property change. Mathematical objects have many properties that do not change: seven is unalterably the square root of forty-nine, the sum of four and three, and prime. The mathematical properties of seven do not change, even if it ceases being my daughter's favorite number. Physical objects do not have eternal, mathematical properties.

We should not consider at least some of the properties in question real properties, anyway. Resnik argues that attempts to refine the notion of property to rule out the changeable properties of mathematics are doomed to failure. For example, one might denigrate all relational properties. Resnik alleges that this would define electrons, which have all their properties relationally, out of existence. "[E]lectrons... have properties only by virtue of their relations to other particles." (Resnik (1997) p 108)

Electrons do have many relational properties, like charge and spin, which are calculated relative to other electrons. They also have calculable mass and velocity, which are more plausibly considered as independent of other electrons. Even if all our knowledge about electrons comes from their relation to other objects, at least some of their properties may still be non-relational. One might be led to believing that all properties of electrons are relational since we have no direct perception of them. We only know of isolated electrons through their relations to observational devices. Still, properties of electrons may be independent of our observations.

The properties of mathematical objects which change are extrinsic, unlike their eternal mathematical properties. Jerrold Katz introduced the notions of intrinsic and extrinsic relations of numbers. "[T]here is no intrinsic change in the number seventeen when someone stops thinking of it and starts thinking of the number eighteen, but only an extrinsic relation between the person and the realm of numbers..." (Katz (1998) pp 136-7)

Some properties, like this epistemic one, are extrinsic. The discovery of Euler's constant does not change it, even though on the most broad notion of properties we can ascribe a property change from being unknown to humans to being known. Similarly, the property of marking Smith's height-in-inches at age 12 is extrinsic to 60. A proper theory of properties might focus only on intrinsic properties, and rule out extrinsic ones.

Lacking a complete theory of properties, I remain agnostic. One might develop a successful theory of properties on which mathematical objects really do have some changeable properties. Still, we can distinguish mathematical and physical objects by the wealth of properties mathematical objects have which do not change. A successful argument for ontic blur must be found elsewhere.

§5.2: Mathematical Objects and Participation in Events

Resnik argues that mathematical objects participate in the events which we describe with their indispensable help. But his resultant argument for ontic blur based on mathematical objects participating in events also fails. Part of the problem is that Resnik conflates the explanation with the event.

If we consider paradigmatically mathematical objects, such as numbers and circles, Resnik's claim that mathematical objects participate in events is clearly false. Resnik considers other objects, of three kinds:

1) Physical functions, such as velocity;
2) Subatomic particles;
3) Fields.

A nominalizing project for functions could negate the possibility of them participating in events. There would be no need to posit the abstract objects that purportedly participate in events. Anyway, the viability of a nominalist project is moot since there are better ways to understand functions like velocity. Just as we distinguish between pure set theory, and set theory with ur-elements, we can distinguish between purely mathematical objects and applied ones. Velocity is an applied mathematical object.

Even if we can dispense with mathematical objects in our explanations of notions like velocity, "We have no reason to think that [such a nominalizing project] can be done with events involving subatomic particles, whose basic features, such as charge, spin and energy level, correspond to no commonsense ideas." (Resnik (1993) p 45)

Nominalizing projects, Resnik implies, may only work when the concepts to be nominalized correspond to commonsense ideas. This suggestion is implausible. If the general public were better educated about subatomic physics, these features would be commonsensical.

Resnik admits that this is the weakest of the three cases. Consider the ball thrown directly upward. Resnik claims that the explanation of the ball stopping its ascent and returning downward requires reference to mathematical objects, including the velocity of the ball, and the vector sum of the upward and downward velocities. Resnik is right that we use the velocity, a function, in the explanation, but it does not participate in the event. The ball participates in the event. The air and the Earth participate in the event. The hand which tosses the ball participates in the event. The velocity participates only in our explanation of the event.

If the elements of scientific descriptions were also elements of the event itself, then Resnik could begin to make his claim, but this is also implausible. We cannot perform scientific explanations of the Big Bang, for example, without using words, I suspect, but there were no words in that physical event.

The second and third examples of mathematical objects participating in events collapse, since quantum particles are, for Resnik, like mathematical objects due to their relations to fields. An electron might be better understood as a manifestation of a field at a point, and the field may be plausibly construed as a distribution of probabilities. "This suggests to me that quantum fields straddle the border between mathematics and physics. Under certain conditions they have 'observable' physical properties, under others they are little different from functions from space-time to probabilities." (Resnik (1997) p 104)

If we construe space-time substantivally, then even if we take a field to be a function from space-time to probabilities, it is no more a mathematical object than the velocity of a moving train. Even if we do not take space-time substantively, the field is located in space and time.

Mark Balaguer agrees that Resnik's position conflates the mathematical properties of an object with the object itself. "I do not see why a full-blown realist about quantum fields and superposition states cannot maintain that while fields can be represented by probability functions, they are not functions themselves." (Balaguer (1999) p 115)

Resnik adduces another example from quantum mechanics as evidence of the mathematical nature of quantum objects. He cites David Bohm's interpretation of a particle's wave function as physically significant, as a force field which guides the particle's trajectory. This wave function splits into parts, with only one part following the particle. "The remaining parts of the wave function/force field are completely undetectable, are causally inert, and have no effects on other particles... Bohm's proposal blurs the distinction between mathematical and physical objects because the vacant parts of the wave function are undetectable and causally inert." (Resnik (1997) p 105)

Mathematical objects share the properties Resnik describes, but this does not make the vacant parts of the wave mathematical objects themselves. They are still parts of the field of a particular particle, and located in space and time, or at least in time, if in superposition. Resnik admits, "But presumably they are located in space-time and thus not fully abstract." (ibid) There is no reason to think they are abstract at all, as long as they are located in space-time.

Resnik is clearly right that quantum particles and fields participate in events. But they are not mathematical objects. He correctly insists that mathematics is relevant in describing fields. But even the indispensable use of mathematics in describing an object is no indication that the object is mathematical. Our best scientific explanation of the behavior of the table in front of me will appeal to its mathematical properties. It is one table, with four legs, of dimensions we can describe geometrically and arithmetically. Even its color properties are describable, at least in part, by wavelengths of light reflected by its surface. The possession of mathematical properties is thus no indication that an object is mathematical, and is no indication of ontic blur. For an object to be mathematical, it must lack physical properties altogether.

Resnik argues that black holes and virtual processes like photon-electron-positron-photon transformations also are like mathematical objects, in being undetectable. These may share undetectability with mathematical objects, but that also does not make them mathematical objects. Resnik says, “It’s not clear that the interiors of black holes or the vacant parts of Bohm’s wave fronts are supposed to be physical in any ordinary sense.” (Resnik (1997) p 107) Still, they are physical, even if extraordinary.

Mathematical objects are empirically undetectable. Some physical objects are, too. If being undetectable by humans were sufficient to make an object mathematical, Resnik’s argument might work. But this criterion is implausible.

In part, Resnik labels quantum particles and fields ‘mathematical objects’ due to his conflation of space-time and mathematical points. If space-time points are mathematical, then fields, which are distributions of intensities over regions of space-time points, could also be construed as mathematical. But the difference between space-time and mathematical points is easily seen in the divergence of Euclidean space and our actual Riemannian space. We could not even understand alternate geometries if there were not an important difference between points in space and geometric points, between lines which extend infinitely in both directions and lines which curve with the shape of the universe. Parallel lines never meet in Euclidean space, maintaining a constant distance between them; lines of space-time can diverge or converge, with the curvature of the universe.

Resnik argues that there are objects which exist in space and time yet are mathematical. They can, therefore, participate in events and there is no good way to draw the line between mathematical and physical objects. This argument relies on the possibility of mathematical objects being located in space and time. If we can show independently that it is impossible for mathematical objects to exist in space-time, then it follows, a fortiori, that they can not participate in events, even if they participate in our descriptions of those events. Thus we have no reason to blur the line between mathematical and physical objects. In the next section, I deny Resnik's contention that mathematics objects exist in space and time.

§5.3: Mathematical Objects and Space-time Location

Resnik poses a series of challenges for the view that mathematical objects are not in space-time, though physical objects are.

> But what is it to be in space-time? To be located in it? To be part of it? To be either? Are space-time points in space-time? Is all of space-time in itself? These are not idle questions. The ontic status of the universal gravitational and electromagnetic fields, prima facie physical entities, as well as that of space-time points, prima facie mathematical entities, turns on how we answer them... Moreover, even quantum particles, such as electrons, widely regarded as paradigm physical objects, pose difficulties for a locationally grounded division between the mathematical and the physical. Where are these particles when they are not interacting with each other? On one interpretation of quantum theory, under some circumstances, these particles are not even located within a finite region of space-time. Then, are they everywhere or nowhere? (Resnik (1993) p 44)

We can extract one major objection to the thesis that mathematical objects can not be located in space-time: the criteria for spatio-temporal location are not clearly defined for a variety of objects, including:

a) Space-time points;
b) Space-time itself;
c) Gravitational and electromagnetic fields;
d) Quantum particles.

Elsewhere, Resnik argues for ontic blur on the basis of the unclear status of

e) Undetectable objects, such as the interiors of black holes.[14]

Resnik implies that without a criterion to determine whether a-e are located in space-time, we can not rule out mathematical objects being included in space-time. The challenge he poses is to find a criterion for spatio-temporal location that will unambiguously place paradigmatically mathematical objects outside of space-time, leave paradigmatically physical objects inside of space-time, and provide reasonable determinations for the hard cases a-e.

Jerrold Katz' abstract/concrete distinction fits the bill. "An object is abstract just in case it lacks both spatial and temporal location and is homogeneous in this respect. An object is concrete just in case it has spatial or temporal location and is homogeneous in this respect." (Katz (1998) p 124)

On Katz' criterion, numbers, pure sets, and geometric figures are abstract, homogeneously lacking spatio-temporal location. Chairs and donkeys and persons are concrete,

[14] See Resnik (1997), p 106.

since they all have at least temporal location. For Katz, objects, like the equator, may also be composite, with both abstract and concrete properties.

Katz' criterion distinguishes physical objects from mathematical objects. But it is based on the spatio-temporal properties of objects. We first have to know whether an object is located in space and/or time to determine whether the object is abstract or concrete. While it looks like Katz's criterion may not help here, we can use it in reverse, starting with our basic intuitions about whether an object is abstract. The challenge is to apply it to the hard cases.

Fields, quantum particles, and undetectable objects, cases c, d, and e, are concrete objects. Even if they are not located at particular positions in space, they have temporal location. Resnik's complaint that they may not have spatial location is moot, since an object need only possess spatial or temporal location, not both, to be concrete.

Resnik argues that some particles are undetectable in principle, and so are causally inert. Since mathematical objects are also causally inert, the undetectable objects are mathematical. Fallacy aside, whatever principle Resnik has in mind links detectability to human perceptual processes and instruments. This is a different principle from the one which grounds the undetectability of mathematical objects.

Resnik calls space-time points, case a, prima facie mathematical objects, repeating the failure to distinguish between pure and applied mathematical objects. Resnik points out that Hartry Field and Geoffrey Hellman, in constructing nominalist systems for mathematics, rely on space-time points, and thus think them nominalistically admissible. But, he argues, the nominalist has to show that space-time points are more epistemically accessible than mathematical objects. Field gives three reasons for thinking so:

1) We can observationally test theories about space-time, as we have tested relativity;
2) Space-time points are a part of space, and thus a part of our physical world - we can even see some;
3) We can identify fields with properties of space-time, making space-time points and regions causal agents. (Field (1982a) pp 68-70; cited in Resnik (1997) p 109)

Resnik argues, in opposition to 1), that we can only apply indirect tests to these theories, like those which would work for mathematical objects. Against 2), he argues that we can only see space-time points if we see the matter within, and that means we would have to accept 3). And 3) only yields space-time regions. We can not attribute the same property to points.

In response to this last argument, the nominalist could appeal to the same kind of rounding-out thesis that the indispensabilist uses to ensure simplicity. From the empirical nature of space-time regions, it is easy enough to argue for points on the basis of mereological methodology. Points could be posited as constituents of regions.

The spatial status of space-time points is puzzling, but Resnik's response does not show that they are not more epistemically accessible than mathematical objects. He merely shows that they are not as tractable as ordinary physical objects. This is no objection to the concrete nature of space-time points, and no argument for ontic blur.

Resnik's appeal to the geometric properties of space-time to establish blur again confounds an object with its properties. "The tendency for physicists to seek structural explanations of the fundamental features of physical reality also undermines the idea that a fundamental ontic division obtains between the physical and mathematical... [P]hysicists have proposed that all of physical reality is an eleven-dimensional space, whose geometric properties give rise to all of the known physical forces." (Resnik (1993) p 46) If reality is an eleven-dimensional space, say, it follows that the parts of the physical world are in space, and thus concrete, according to Katz' criterion. Resnik's blur is not established. We might, truly, give up the notion that the elements of our physical space are most correctly described in the language of bodies. Resnik's further step, to argue that this makes the parts of space mathematical objects, is unwarranted.

As for the remaining case b, whether space-time itself is located within space-time, if there is any object to which Resnik refers by "space-time itself," is no more puzzling than whether any object contains itself. This may be a question for theorists of material constitution. It is no argument for blur.

§5.4: The End of Blur

Resnik attempts to establish ontic blur in order to support the claim that there is no evidentiary distinction between mathematical and physical objects, and support the structuralist's contention that our perception of concrete arrangements leads to knowledge of abstract objects. Resnik's arguments assimilating mathematical and physical objects fail. Mathematical objects do not change their properties, they do not participate in events, and they are not located in space-time. While some physical objects have properties that mathematical objects have, like not being directly detectable by humans, they are not mathematical objects.

Resnik does not extend his characterization so far as to call commonsense physical objects mathematical. "While subatomic particles occupy the vague region between mathematics and physics, tables and chairs are unquestionably not mathematical objects." (Resnik (1997) p 265) This clarity about the concreteness of ordinary physical objects clashes with his more general holism. Really, for the holist and his blur, terms like 'mathematical object' and 'physical object' are ill-formed, and there are just objects with some properties traditionally labeled 'concrete' and others traditionally labeled 'abstract'.

Balaguer noticed the conflict. "What I find puzzling in Resnik's view is not so much his blurriism as the fact that he embraces blurriism together with the thesis that abstract objects exist 'outside space and time.' For it seems to me that the thesis of non-spatio-temporality brings with it a very clear abstract-concrete distinction." (Balaguer (1999) p 114)

The abstract/concrete distinction best distinguishes mathematical from physical objects. The best explication of this distinction is in terms of spatio-temporal characteristics of concrete objects. The lack of spatio-temporal characteristics for mathematical objects may explain, in part, why they lack causal powers, but we must not characterize everything which lacks causal powers as mathematical.

§6: The End of Structuralism

A realist philosophy of mathematics must provide an account of human access to abstracta. Insofar as the structuralism I have considered provides any answer to this question, it appeals to the use of mathematical objects, structures, to account for our knowledge of the physical world. This is the indispensability argument, and it suffers the Unfortunate Consequences.

Aside from its inability to answer the main epistemic challenge to mathematical realism, Balaguer has argued that structuralism fails to solve the problem of multiple reductions for the natural numbers to sets, its motivating program. The structuralist offers unique structures which can be instantiated by various sequences of objects, the von Neumann sets and the Zermelo sets, say. But the structures themselves may suffer from a similar indeterminacy. "[T]here may be multiple structures that satisfy all the desiderata for being the natural number sequence and differ from one another only in ways that no human being has ever imagined..." (Balaguer (1999) p 117; see also Balaguer (1998), Chapter 2.)

Resnik argues that taking mathematical objects as positions in patterns is not intended as an ontological reduction. Since most formulations of number theory do not have individual variables for sets, they can not assert that there is a number sequence. They treat numbers but not the sequence or structure. In general, a theory can not require its universe of discourse to contain itself, though we can extend a theory to include its (sub-)self. Just as numbers can be both nouns and adjectives, and when we explicate the numbers, we paraphrase one of the uses in terms of the other, so we can see patterns both as individuals and not, and paraphrase away the individual uses. Resnik's response amounts to a denial of indeterminacy among structures, since there are no facts concerning which sets are the numbers, or which structures are the number-theoretic structures, about which to be indeterminate.

We can deny that there is a fact of the matter for some question by excluding it from the field of our truth predicate, or by weakening our logic. But we can also, more strongly, just bar the question. Resnik denies that sentences such as "numbers are sets" have truth value, but we can not banish 'set' from our language. Resnik prefers to restrict logic so that excluded middle does not apply generally. In his defense, he argues that our commitment to many disjunctions, like those involving fictional or vague terms, is weak. In the cases Resnik cites, though, we do want to banish the terms, rather than restrict excluded middle. We do not adjust excluded middle because we are puzzled about the status of "Nemo the clownfish is cute." "Numbers are sets," on the other hand, is false.

Balaguer agrees. "If we look at mathematical practice *as a whole*, it is, I think, apparent that there are very definite facts of the matter about these questions: mathematicians think that numbers are *not* identical with sets and that 2 is not identical with {{0}}" (Balaguer (1999) p 116)

Resnik argues that his structuralism is independent of the indispensability argument, but it depends on indispensability if it is to be of any use in answering the epistemic challenge to mathematical realism. Structuralism suffers the Unfortunate Consequences, and the structuralist fails to establish ontic blur in order to use our perceptions of concrete patterns as evidence for abstract mathematics. Structuralism does not even solve its motivating problem. We may still hold this much of structuralism, that all that matters to mathematicians are positions in structures, as the theorems define them, without adopting structuralism in the philosophy of mathematics.

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