E Pluribus Putnams Unum: Hilary Putnam and the Indispensability Argument

Russell Marcus

The Graduate School and University Center of the City University of New York

June 2006

~9,400 words

Abstract:

In this paper, I argue that the indispensability argument unifies all of Hilary Putnam's diverse work in the philosophy of mathematics. I also argue that his version of the indispensability argument does not justify knowledge of mathematical objects.

I first characterize indispensability arguments in the philosophy of mathematics, distilling them to four essential characteristics. I show that there is a sufficient but not necessary condition on being an indispensabilist, which may be used as a quick test for reliance on the argument. I apply this test to Putnam to show his reliance on the argument through his work. Then, I show in detail how each of four Putnams (the deductivist, the modalist, the realist, and the anti-realist) adopts the indispensability argument. I argue that Putnam's success argument, his version of the indispensability argument which does not rely on Quine's holism, does not establish mathematical knowledge. The objects it purports to yield are not truly mathematical, and the success argument rest on a false premise.

§1: Introduction: the Indispensability Argument

Hilary Putnam may be both lauded and criticized for working on a variety of competing positions, especially in the philosophy of mathematics. Approaching Putnam's work can leave philosophers in the dizzying position of trying to grasp a moving target. In this paper, I argue for a unification thesis: Putnam's shifts in the philosophy of mathematics are mostly cosmetic, and reliance on mathematical empiricism in the guise of one or another form of the indispensability argument is a deep underlying theme through his work.

The indispensability argument, in its most general form, alleges that our knowledge of mathematics is justified by our knowledge of empirical science. Putnam held at least two specific versions of the indispensability argument. The first is Quine's, which Putnam echoes.[1] The second is his own success argument. Neither Quine nor Putnam formulate a detailed argument, though Putnam comes closer.[2] The following essential characteristics of any indispensability argument for mathematical realism apply to both versions, as well as others which Putnam did not defend.

EC.1: Naturalism: The job of the philosopher, as of the scientist, is exclusively to understand our sensible experience of the physical world.
EC.2: Theory Construction: In order to explain our sensible experience we construct a theory or theories of the physical world. We find our commitments exclusively in our best theories.
EC.3: Mathematization: We are committed to the truth of some mathematical statements

[1] Putnam 1956 presents an indispensability argument which is not clearly distinct from Quine's. Putnam 1971 defends Quine's argument, in places.

[2] One can find elements of Quine's indispensability argument in Quines 1939, 1948, 1951, 1955, 1958, 1960, 1978, and 1986b. See below, especially §5, for references to Putnam's indispensability arguments.

since they are essential to that best theory.[3]

EC.4: Subordination of Practice: Mathematical practice depends for its legitimacy on empirical scientific practice.

For the indispensabilist, the justification for any mathematical knowledge must appeal to an account of our sense experience. EC.1 and EC.2, for example, rule out appeal to mathematics independently of science for justification of mathematical claims. EC.4, while implicit in the other characteristics, emphasizes the relationship between mathematics and empirical science for the indispensabilist.

For the purposes of this paper, I take mathematical realism as the position that some mathematical claims are non-vacuously true. Then, there is an easy way to tell that a mathematical realist is an indispensabilist: if he argues that the theory of relativity shows that Euclidean geometry is false, he links mathematical truth with empirical evidence.[4] That is the core of the indispensability argument. One can appeal to empirical evidence for mathematical statements without being an indispensabilist, but only if one abandons mathematical realism. Putnam consistently takes this telltale position regarding Euclidean geometry despite exploring various positions in the philosophy of mathematics: Deductivism (1967a), Modalism (1967b, 1975a), Realism (1971, and 1975a again), Anti-realism (1980, 1981, 1994).

Given that the indispensabilist attempts to establish mathematical realism, there seems to be a tension between my ascription of indispensability and Putnam's three non-realist positions. In the case of anti-realism, this tension is merely apparent. Putnam relies on indispensability to

[3] Mathematization may also be cast in terms of commitments to the existence of objects. I focus on truth, here, for consistency with Putnam's concerns and emphasis.

[4] This is a sufficient, though not a necessary, condition for being an indispensabilist.

generate as much realism for mathematics as he generates for anything else. His anti-realism, which he labels "internal realism," denies that we can establish transcendent realism, or truth, or reference, for any domain, including mathematics and physical science. Empirical and mathematical justifications remain linked.

Putnam's modalist claims are subordinate, in his presentation, to the claim that modalism is equivalent to realism, and also to a stronger nihilist claim, that there are no correct philosophies of mathematics. Here again, Putnam's position only conflicts with indispensability realism on the surface. Putnam's modalist accepts evidence for mathematical statements, and this evidence is empirical.

As for deductivism, Putnam's underlying argument is for mathematical realism based on indispensability, as I shall show. First, I clarify my claim about the telltale position and Putnam's consistent adoption of it.

§2: The Telltale Position

In an early paper, Putnam argues that we should adopt three-valued logic because of its utility in accommodating quantum mechanics. Putnam accepts the demand that we need to be shown the broad applicability of this logic. "This objection, however, cannot impress anyone who recalls the manner in which non-Euclidean geometries were first regarded as absurd; later as mere mathematical games; and are today accepted as portions of fully interpreted physical hypotheses." (Putnam 1957: 169)

Every one agrees that Riemannian geometry is a portion of a fully interpreted physical hypothesis, relativity theory. Putnam reveals his indispensabilism by implying that non-

Euclidean geometries are mere mathematical games until they become portions of physical theory. More insidiously, he implies that geometry is a physical hypothesis. Though Putnam later rejects the proposal to adopt Reichenbach's three-valued logic, the seeds of his indispensabilism are evident.

Stronger statements of the telltale position are ubiquitous in Putnam's later work.

> If there are any paths that obey the pure [Euclidean] geometrical laws (call them 'E-paths'), they do not obey the principles from physics... Either we say that the geodesics are what we always meant by 'straight line' or we say that there is nothing clear that we used to mean by that expression. (Putnam 1968: 177)

> The received view is that the temptation to think that the statements of Euclidean geometry are necessary truths about actual space just arises from a confusion. One confuses, so the story goes, the statement that one can't come back to the same place by traveling in a straight line (call this statement 'S'), with the statement that S is a theorem of Euclid's geometry... I find this account of what was going on simply absurd. (Putnam 1974: ix)

> If space were Euclidean, doubtless the distinction between 'mathematical' and 'physical' geometry would be regarded as silly... Euclidean geometry is false - false of *paths in space*, not just false of 'light rays'. (Putnam 1975a: 77-8)

> Unless one accepts the ridiculous claim that what seemed *a priori* was only the *conditional* statement that *if* Euclid's axioms, then Euclid's theorems (I think that this is what Quine calls 'disinterpreting' geometry in 'Carnap and Logical Truth'), then one must admit that the key propositions of Euclidean geometry were *interpreted* propositions ('about form and void', as Quine says)... (Putnam 1976: 94)

> Suppose someone had suggested to Euclid that this could happen: that one could have two *straight* lines which are perpendicular to a third *straight* line and which *meet*. Euclid would have said that it was a necessary truth that this couldn't happen. According to the physical theory we accept today, it *does* happen. (Putnam 1981: 83)

Putnam's concern with Euclidean geometry is often not its falsity but the effect of the shift to non-Euclidean geometry on a priori knowledge. Traditionally, Euclidean geometry was taken to be known a priori. Apriority was taken to imply necessary truth. If geometric principles turn out false, we have a conceptual problem.

The traditional solution distinguishes between pure and applied mathematics. The adoption of Riemannian geometry as the framework for relativity theory does not refute Euclidean geometry, but shows its inapplicability to physical space. Still, both Euclidean and Riemannian geometries can be known a priori. On this account, which Putnam calls absurd in the second quote above, Euclidean theorems hold necessarily of Euclidean space. When Newtonian mechanics was replaced by a theory based in Riemannian space, no necessary entailments were threatened. Only the empirical commitments of the theory were changed.

The traditional solution does not sever the link between mathematics and science ad hoc, to maintain a priori knowledge. The distinction is independently plausible. Only the Kantian assumption that we have a priori knowledge of the physical universe supports the belief that our knowledge of mathematics was based on a priori knowledge of physical space. Whereas Hume may have stirred Kant from his dogmatic slumbers, Kant apparently did not awaken completely. Hume taught us that matters of fact could not hold a priori, including knowledge of the structure of physical space.

In part, Putnam defends his claim that an a priori and necessarily true statement turned out to be false by referring to the inconceivability of non-Euclidean geometries to pre-relativity thinkers. Inconceivability is a strong claim. More plausibly, one could claim that they merely failed to conceive of non-Euclidean geometries. Our intuitions about mathematical spaces and structures are constantly being extended. The resulting mathematics is not justified by finding physical correlates, but by its fruitfulness, among other factors. Failure to consider consistent but unintuitive spaces is no argument against their mathematical existence. The traditional a priori account need not be dismissed.[5]

Whatever the motivation for Putnam's adoption of the telltale position, he consistently holds it, illustrating my unification thesis. I proceed to develop this thesis in detail for each of the four Putnams, showing how each of them shares the four Essential Characteristics.

§3: Putnam's Deductivism

The case for the ubiquity of Putnam's indispensabilism is hardest to make for his deductivism. He refers uncritically to the pursuit of pure mathematics and appeals to non-empirical considerations when attempting to specify a standard model. Still, indispensability is present.

Putnam explores deductivism as a response to problems with Russell's *Principia* logicism. He argues that logicism best accounts for the application of mathematics, but recognizes three problems. First, Cantor and Gödel showed that we can not talk about all sets, or all natural numbers, due to diagonalization arguments. Second, the Löwenheim-Skolem theorem shows that we can not determine a standard model from within a mathematical theory strong enough to serve the needs of mathematics. Third, the independence of the continuum hypothesis shows that if we could determine a standard model for set theory, we would not know whether to add the continuum hypothesis or its negation. Logicism commits to a unique solution to a question that should be left open.

---

[5] Another way to resolve the problem is to sever the link between apriority and necessary truth. One could adopt a fallibilist apriorism, which separates issues about how we come to justify a statement from questions of truth or necessity. See §4, below.

In lieu of logicism, Putnam suggests returning to Russell's earlier deductivism, which Russell had abandoned for logicism's account of application and the possibility of specifying a standard model. Since, Putnam argues, Russell erred about our ability to specify a standard model, logicism loses its draw and we should revert to the earlier position, and devise an alternate account of application. Pursuing deductivism, Putnam proposes to take mathematics as the study of the consequences of mathematical axioms, using model theory. He reinterprets mathematical statements as referring to the possibility of a model for those statements. Any theorem of any mathematical theory T of the form "$\exists x \mathscr{F} x$" really means that if there is a model M of T, then there is something in M which is $\mathscr{F}^M$. Deductivism does not turn mathematics into logic in the logicist's sense, but it is logic in a broader sense which includes the set theory we need to construct models to determine the consequences of axioms.

Putnam only appeals to deductivist interpretations for mathematical statements which can not be rendered in first-order logic plus empirical language. Consider, as Putnam does, a room with two apples on the desk, two apples on the table, and no apples on both the table and desk, or elsewhere. We can infer that there are four apples in the room with merely first order logic in addition to the empirical premises and with no need for mathematics. Putnam has adopted one of the Essential Characteristics: EC.4, Subordination of Practice. Mathematics only requires and receives justification when it is useful to empirical science.

Discussing Kant, Putnam laments the impossibility of determining a standard model on the basis of empirical evidence. We could fix a standard model on the basis of intuitive chronometry, or geometry, if we took those principles to be known a priori. If we could guarantee the existence of a standard model, then we could know mathematical truths based on this intuitive grasp of the theorems of mathematics.

But, Putnam continues, Kant was wrong about our a priori grasp of geometry and chronometry, and there are non-standard models of our mathematical theories. Putnam here commits to EC.2, Theory Construction. Empirical evidence is insufficient to generate a standard model. By limiting his appeals to the models (standard or non-standard) and neglecting to include auxiliary mathematical evidence which could help fix the standard model, he again reveals his commitment to Subordination of Practice. A non-indispensabilist can accept non-empirical evidence for the standard model.

Putnam's preference for logicism, in his argument for deductivism, is really an expression of EC.1, Naturalism: the primary goal of the mathematician is to provide tools to assist with our understanding, or explanation, of the physical world. Logicism could explain the applicability of mathematics to the empirical world, by assimilating mathematics to logic, which applies to all possible states of the world. Putnam's deductivist has adopted three of the four Essential Characteristics of indispensability arguments.

In the course of resolving a problem for deductivism, Putnam seems to appeal to mathematics beyond the indispensabilist's yield. The deductivist has a prima facie difficulty selecting certain derivations, say the finite ones, from a given mathematical theory. Mathematicians often study only restricted areas like finite number theory. Since deductivism interprets mathematics as the study of entailments from axioms, it can not give priority to any sub-class of consequences.

In response, Putnam attempts to show that there is no need to make such distinctions formally, since the notions involved, like finitude, are relative to a broader model. We can not construct an absolute notion of 'finite', or of a standard model, within a formal mathematical theory. Our interest in finite structures, or in only the standard model, is not based on any mathematical priority of those portions of our larger mathematical theory.

This blanket legitimation of all consequences of a set of mathematical axioms appears to avoid the indispensabilist's inability to justify mathematical results which are unapplied by rejecting EC.4. But, Putnam continues, "None of this really presupposes the *existence*, in a non-hypothetical sense, of any models at all for anything, of course!" (Putnam 1967a: 23) That is, deductivism does not generate any commitments, including ones to mathematical objects. Putnam is not justifying more mathematics than the indispensabilist; he is remaining agnostic on the question of mathematical existence. Still, he does have some existence questions in mind. Putnam considers models with empirical domains, and he characterizes finitude and a standard model using set theory with empirical elements.

Here is one last consideration in favor of my claim that indispensabilism underlies Putnam's deductivism. Putnam rejects Hilbert's distinction between real mathematical statements, which admit of constructive proofs, and ideal statements, which can not be proven constructively. He prefers a distinction between which statements can be applied and which can not. Putnam wants to turn Hilbert into a kind of indispensabilist, limiting mathematical truth only to those statements which can be applied to empirical theory.

The indispensabilist thread to Putnam's deductivism appears in his reliance on empirical evidence for mathematical statements. Even when he looks to broaden the sources of evidence for mathematics beyond those to which the indispensabilist is entitled, he rejects the mathematician/logician's notions of 'property' and 'relation' as arcane, and arbitrary, and seeks to generate a "natural" concept of properties and relations. Putnam's rejection of deductivism in favor of modalism, which I consider next, is based in large part on considerations even more in line with indispensability.

§4: Putnam's Modalism

In both of Putnam's modalism papers, he takes the telltale position on geometry. In 1967b, he notes that the development of non-Euclidean geometries showed that the axioms of Euclidean geometry were not truths. "The price one pays for the adoption of non-Euclidean geometry is to deny that there are *any* propositions which might *plausibly* have been in the minds of the people who believed in Euclidean geometry and which are simultaneously clear and true." (Putnam 1967b: 50)

Putnam does not strongly defend modalism in either paper. He rejects the existence of a correct position, arguing that the modal account is equivalent to a realist one. Putnam characterizes the equivalence between modalism and realism as 1) definability of the primitive terms of each theory in the primitive terms of the other; and 2) deducibility within each theory of the other. He considers a (fantastic) counterexample to Fermat's theorem. It would be describable in object-realistic terms: the existence of four positive integers x, y, z, and n (where $n > 2$), such that $x^n + y^n = z^n$. We could also write it as schema of pure first-order modal logic: $\Box[AX(S, T) \supset \sim Fermat(S, T)]$, where 'AX' represents the conjunction of mathematical axioms required to generate the numbers required for the counter-example, 'S' and 'T' are dummy predicate letters, and 'Fermat' represents the claim that there are no solutions to the given schema. Both sentences, Putnam claims, assert the same fact, but modalism requires no objects, merely describing entailments, whereas realism requires a vast universe of mathematical objects. "'Numbers exist'; but all this comes to, for mathematics anyway, is that 1) ω-sequences are possible (mathematically speaking); and 2) there are necessary truths of the form 'if α is an ω-sequence, then...' (whether any *concrete* example of an ω-sequence exists or not." (Putnam 1967b: 49, emphasis added)

There are four claims on the table: nihilism (there is no correct position in the philosophy of mathematics); equivalence (there are equivalent modalist and object realist formulations of all mathematical claims); realism; and modalism. Putnam bases nihilism on equivalence, but a weaker claim that there are multiple correct positions is more plausible.

Still, the equivalence claim seems false. Putnam argues that we can take possibility as primitive and derive sets from it. Hartry Field argues that there is no acceptable modal operator to do the work that Putnam needs to establish equivalence.[6] Burgess and Rosen have other complaints about Putnam's project. "The technical details of the modal reconstrual he proposed are of no continuing interest, among other reasons because he did not deal with mixed, mathematico-physical language..." (Burgess and Rosen 1997: 201)

The failures of Putnam's modalism need not concern us. The question here is how modalism relies on indispensability. The answer involves Putnam's demand for concrete models. While the modal claim, summarized as "sets are permanent possibilities of selection," makes the models he uses abstract in a sense, these are models whose domains contain only concrete objects. Putnam relies on concrete models, like inscriptions of a graph. "In constructing statements about sets as statements about standard concrete models for set theory, I

[6] See Field 1988, especially §6-§8.

did not introduce possible concrete models (or even impossible worlds) as objects. Introducing the modal connectives...is not introducing new kinds of objects, but rather extending the kinds of things we can say about ordinary objects and sorts of objects." (Putnam 1967b: 58-9)

In the later modalism paper, Putnam also explicates possibility using concrete elements. "A statement to the effect that for every number x there exists a number y such that F(x, y), where F(x, y) is a recursive binary relation, can be paraphrased as saying that it is not *possible* to produce a tape with a numeral written on it which is such that if one *were* to produce a Turing machine of a certain description and start it scanning that tape, the machine would never halt." (Putnam 1975a: 71-2) Turing machines and their tapes are concrete objects, even if their programs are abstract.

Also, Putnam discusses modalism in the context of quasi-empiricism; this is the paper in which he introduces 'Martian Mathematics'. In Putnam's scenario, we discover alien mathematicians with a relaxed standard, relative to our own, for the acceptance of mathematical statements. The Martians accept statements which have not been proven, but only confirmed: counter-examples have not been discovered and the statements seem to cohere with a larger body of accepted results.

Putnam calls the Martian methodology quasi-empirical, because the accepted theorems are defeasible. His usage reveals a terminological confusion. A statement's being defeasible is not necessarily indicative of its being empirical. On a fallibilist apriorism, statements believed on the basis of a priori considerations may be ceded on the basis of further a priori reflection. Putnam's characterization of empirical science rules out this position. According to Putnam's criterion, defeasibility makes mathematics quasi-empirical, even if our mathematical beliefs are known independently of experience.

Putnam's classification relies on his independent criticism of the a priori. That argument focused on how beliefs which had been taken to be a priori turned out to be false. He wrongly presumes that apriority entails necessary truth. Recognizing that we need a fallibilist account of mathematics, he concludes that it must be quasi-empirical.

Putnam mistakes the revisability of mathematics to entail that mathematical claims are not known a priori. He then assimilates mathematics to empirical science. Mathematical necessity thus becomes empirical necessity. Mathematical truth becomes empirical truth. Putnam becomes an indispensabilist.

The defeasibility criterion for quasi-empiricism slightly revises his earlier discussion of the (sort of) empirical nature of mathematics, based on the availability of viable competitors. Here too, Putnam confuses empiricism with fallibilism.

> [T]he chief characteristic of empirical science is that for each theory there are usually alternatives in the field, or at least alternatives struggling to be born. As long as the major parts of classical logic and number theory and analysis have no alternatives in the field - alternatives which require a change in the axioms and which effect the simplicity of total science, including empirical science, so that a choice has to be made - the situation will be what is has always been. We will be justified in accepting classical propositional calculus or Peano number theory not because the relevant statements are 'unrevisable in principle' but because a great deal of science presupposes these statements and because no real alternative is in the field. Mathematics, on this view, does become 'empirical' in the sense that one is allowed to put alternatives into the field. (Putnam 1967b: 50-1)

Given Putnam's odd characterizations of the empirical, one might think that the difference between Putnam's allegation that mathematics is empirical (or quasi-empirical) and apriorism about mathematics is merely terminological. Consider further examples of mathematical claims which Putnam says were acquired empirically, or quasi-empirically: the postulation of a correspondence between real numbers and points on a line, well before the construction of reals out of rationals; the introduction of infinitesimals before epsilon-delta methods in the calculus were developed; Zermelo's use of the axiom of choice. Putnam claims that all of these were originally justified based on their success, fertility, and application, before formal proofs were generated.[7] He takes those criteria to be indicative of empirical (or at least quasi-empirical) justification. "The fact is that *we* have been using quasi-empirical and even empirical methods in mathematics all along..." (Putnam 1975a: 64)

Calling mathematics empirical on the basis of defeasibility, or the presence of viable competitors, does not entail that it relies on empirical evidence, in a traditional sense. One may call it whatever one likes.

Moreover, consider Putnam's argument that mathematics is empirical because the acceptance of some mathematical statements is based on experimentation. The axioms of choice and replacement are not proven, but shown mathematically useful. Euler discovered that $\sum 1/n^2 = \pi^2/6$ through analogical reasoning, positing the equivalence of two terms on the basis of structural similarities, before he had a formal proof.

> [N]o mathematician doubted that the sum of $1/n^2$ was $\pi^2/6$, even though it was another twenty years before Euler had a proof. The similarity of this kind of argument to a hypothetico-deductive argument in empirical science should be apparent: intuitively

---

7 Putnam's use of 'success' is ambiguous, referring sometimes to success within mathematics, and thus not indispensabilist, and sometimes to success within empirical theory. "The real justification of the calculus is its *success* - its success in mathematics, and its success in physical science." (Putnam (1975a) p 66) This ambiguity also reflects Putnam's failure to distinguish revisable a priori beliefs from empirical ones.

> plausible though not certain analogies lead to results which are then checked 'empirically'. Successful outcomes of these checks then reinforce one's confidence in the analogy in question. (Putnam 1975a: 68)

Putnam's allegation that these results are checked empirically seems unjustified. Euler checked the results by calculating the series $1/n^2$ for finite results; Putnam says he checked them up to n = 30. There is nothing empirical about this. We can call it an experiment, in the same way one might call adoption of the axiom of choice an experiment. Then there are a priori experiments.

It looks as if Putnam's claim that mathematics is empirical, or quasi-empirical, may be merely misleading uses of those terms. But when we examine the details of the kind of evidence Putnam's modalist/quasi-empiricist accepts, we see that he really does allow empirical evidence. In the beginning of this section, I provided examples, especially Putnam's reference to concrete models. Consider also that when Putnam talks about the success of mathematical experiments, he focuses on empirical success. "Today it is not just the axiom of choice but the whole edifice of modern set theory whose entrenchment rests on great success in mathematical application - in other words, on 'necessity for science'." (Putnam 1975a: 67)

Putnam confuses reliance on factors like success and fertility with empirical justification. In constructing empirical theory, we look to success and fertility in explaining sense experience. In constructing mathematical theory, we look to success and fertility within mathematics. This does not mean that the difference between Putnam's quasi-empiricism and apriorism is merely terminological, even if the root of Putnam's adoption of empirical evidence in mathematics is this confusion.

This discussion of quasi-empiricism has brought me away from Putnam's modalism for two reasons. Putnam's later discussion of modalism arises within his discussion of quasi-empiricism, although he leaves the connection obscure. Also, quasi-empiricism again evinces Putnam's indispensabilism, by admitting empirical justification for mathematics.

The main distinction between Putnam's modalism and his deductivism is his claim that we can fix a standard model within the modal picture. He asks us to accept it on faith that this can be carried out. Since his preference for deductivism over logicism was largely based on our inability to fix a standard model, Putnam could return to logicism, with the emendation that we can fix the standard model modally. The equivalence claim, and his concomitant nihilism, lead Putnam to give up on a correct philosophy of mathematics. Still, the equivalence claim is that modalism is equivalent to realism, and so he does, in a way, return.

Putnam's equivalence claim lessens the oddity of his reliance on indispensability in his modalism. His anti-realist claims (e.g. "The modal logical picture shows that one doesn't have to 'buy' Platonist ontology..." (Putnam 1975a: 72)) presuppose an equivalence with realism. In the later paper, Putnam also presents his success argument for realism in mathematics, which I evaluate in §7.

§5: Putnam's Realism

That Putnam relies on indispensability for his mathematical realism needs little defense. But, in places he seems to make additional claims for a traditional (i.e. non-indispensabilist) mathematical epistemology. "There are *two* supports for realism in the philosophy of mathematics: *mathematical experience* and *physical experience*." (Putnam 1975a: 73) He describes mathematical experience on analogy with theological experience, independent of empirical evidence. In this section, after contrasting Putnam's indispensability argument with Quine's version, I show that such claims are misleading, and that Putnam's realism is fully indispensabilist, not traditional.

A paragraph after the above quote, Putnam makes his indispensabilism clear.

> If this argument [from mathematical experience] has force, and I believe it does, it is not quite an argument for mathematical realism. The argument says that the consistency and fertility of classical mathematics is evidence that it - or most of it - *is true under some interpretation*....The interpretation under which mathematics is true has to square with the application of mathematics *outside* of mathematics. (Putnam 1975a: 73-4)

Incidentally, I think that the qualification above ("most of it") is an indication that Putnam recognizes that his realism suffers from the indispensabilist's rejection of pure mathematics. He recognizes this more explicitly, earlier.

> Sets of a very high type or very high cardinality (higher than the continuum, for example), should today be investigated in an 'if-then' spirit. One day they may be as indispensable to the very *statement* of physical laws as, say, rational numbers are today; then doubt of their 'existence' will be as futile as extreme nominalism now is. But for the present we should regard them as what they are - speculative and daring extensions of the basic mathematical apparatus. (Putnam 1971: 347)

Putnam's indispensability argument has strong affinities with Quine's. "This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes." (Putnam 1971: 347)

Quine's indispensability argument rests essentially on both his procedure for determining the ontic commitments of theories and his holism.[8]

(QI) QI.1: We should believe the theory which best accounts for our empirical experience.
QI.2: If we believe a theory, we must believe in its ontic commitments.
QI.3: The ontic commitments of any theory are the objects over which that theory first-order quantifies.
QI.4: The theory which best accounts for our empirical experience quantifies over mathematical objects.
QI.C: We should believe that mathematical objects exist.

In early work, Putnam approved of Quine's holism, which led him naturally into indispensability. "I should like to stress the monolithic character of our conceptual system, the idea of our conceptual system as a massive alliance of beliefs which face the tribunal of experience collectively and not independently, the idea that 'when trouble strikes' revisions can, with a very few exceptions, come anywhere." (Putnam 1962: 40)

Differences between Quine and Putnam emerge slowly over the years. Putnam abandoned Quine's commitment to a single, regimented, best theory. He assumed a realist stance about truth in science, in contrast to Quine's view of truth as mainly a device for semantic ascent. Putnam's emphasis on scientific truth led to his success argument, his independent version of the indispensability argument. Also unlike Quine, Putnam argued that mathematics is indispensable to correspondence truth, since it demands relations, and formal logic and semantics, in order to formalize metalogical notions like derivability and validity.

The anti-realist Putnam, who I discuss in the next section, cites two 'modifications' of Quine's indispensability argument. The first is the addition of combinatorial facts to sensations,

---

[8] The holism in question here is confirmation holism, and not Quine's stronger semantic holism, which is mainly irrelevant to the indispensability argument.

as desiderata of theoretical construction. "[T]he idea that what the mathematician is doing is contributing to a scheme for explaining sensation just doesn't seem to fit mathematical practice at all. What does the acceptance or non-acceptance of the Axiom of Choice... have to do with explaining sensations?" (Putnam 1994: 504)

As for the other, "The second modification I propose to make in Quine's account is to add a third non-experimental constraint to his two constraints of 'simplicity' and 'conservatism'... The constraint I wish to add is this: *agreement with mathematical 'intuitions'*, whatever their source." (Putnam 1994: 506)

The acceptance of mathematical intuitions looks like a break not only with Quine, but with earlier Putnam. Putnam seems to have softened toward mathematical intuition and abandoned indispensability. But this appearance is misleading, since Putnam still understands intuition as essentially empirical. Consider the quasi-empiricism which admits combinatorial facts, truths of number theory, geometry, and set theory, as facts to be explained by scientific or mathematical theory.[9] If these combinatorial facts are not posits to explain sensations, as the indispensabilist has it, then where do they come from? How do we know which facts need explaining? How do we separate the truths from the falsehoods? To answer these questions, Putnam posits mathematical intuitions, and argues that they lead us to mathematical truth.

(PI) PI.1: We have intuitions about the truth of mathematical statements, of combinatorial facts.
PI.2: These intuitions are justified quasi-empirically.
PI.3: Quasi-empirical justifications yield truth.
PI.C: So, mathematical statements are true, and justified.

---

[9] Putnam's claim that mathematics is quasi-empirical first appears in his modalist phase, but he carries it through the rest of his work.

Putting aside worries about Putnam's motives for quasi-empiricism, why should we believe that quasi-empirical justifications yield truth? Whether PI is an indispensability argument or an abandonment of the indispensabilist's naturalism, depends on the nature of these quasi-empirically justified intuitions. If they are derived from solely empirical experience, and the quasi-empirical facts are really empirical facts, then Putnam is just repackaging mathematical empiricism with new labels, as he did in his modalist phase.

Putnam's faculty of intuition is not like those of writers who do countenance an independent mathematical epistemology. It is not Gödelian insight, which Putnam rejects as mysticism. Neither is it Jerrold Katz's notion of reason, which Putnam neglects. It is not even Platonic formal acquaintance. Rather, Putnam argues that we establish our mathematical intuitions quasi-empirically, by our attempts to understand the empirical world. Their justifications come from science, often inductively. Our knowledge of mathematics is on this account empirical.

Putnam's discussion of mathematical induction demonstrates his empirical grounding of mathematical intuition. "The principle of mathematical induction, for example, bears the same relation to the fact that when a shepherd counts his sheep he always gets the same number (if he hasn't lost or added a sheep, and if he doesn't make a mistake in counting) no matter what order he counts them in, that any generalization bears to an instance of that generalization." (Putnam 1994: 505)

Putnam uses this example to justify his quasi-empiricism by blurring the line between mathematical reasoning and empirical reasoning. But it is wrong. The relationship of a mathematical generalization to an instance is immediate. The use of mathematical induction in a specific mathematical case yields a proof. In contrast, the relation between a mathematical generalization like the principle of mathematical induction and an empirical instance is mediated by the caveats provided by Putnam, and also by a Humean Principle of Uniformity of Nature, that sheep do not spontaneously generate or disappear, for example.

Putnam only appears to admit mathematical intuitions. They are empirical, even if he calls them mathematical, or quasi-empirical. "[I]t is not clear how mathematical 'intuitions' do [constitute a link between acceptability and truth], if at bottom they are just generalizations from the finite on the basis of human psychology, reified forms of grammar, and so on." (Putnam 1994: 507)

Putnam's overtures to pure mathematics, combinatorial facts, and mathematical intuition, in both his realist and anti-realist phase, are at root just more mathematical empiricism, more indispensabilism.

§6: Putnam's Anti-realism

Putnam's move toward anti-realism from the late 1970s on reflected the growing differences with Quine. For example, he accepted that there are a priori truths.[10] "Not every statement is both true and false," serves as an example. "At least *one* statement is *a priori*, because to deny that statement would be to forfeit rationality itself." (Putnam 1979: 129)

Putnam's acceptance of a priori statements was not, as it might seem, a move toward a more substantial realism than Quine's. His a priori statements are relative to a body of

---

[10]Putnam uses 'a priori' as a modifier of statements, rather than as a description of how one justifies statements. He thereby seems not to use it as an epistemic notion. I shall follow him, for the purposes of discussing his work, though this usage leads to confusion.

knowledge which may be swapped for an alternative body with different a priori truths.[11] Permissible shifts in what we think is rational alter the class of a priori statements.

I call Putnam's later position anti-realism. Putnam's use of 'internal realism' is misleading, as he rejects claims which are central to realism in mathematics, and in ethics and science. These claims, elements of which Putnam calls metaphysical realism, are that we can assert language-independent, or conceptual-scheme-independent, truths. Putnam alleges that even our best scientific theory is interest-dependent.

> If there are many ideal theories (and if 'ideal' is itself a somewhat interest-relative notion), if there are many theories which (given appropriate circumstances) it is perfectly rational to accept, then it seems better to say that, in so far as these theories say different (and sometimes, apparently incompatible) things, some facts are 'soft' in the sense of depending for their truth value on the speaker, the circumstances of utterance, etc. (Putnam 1980: 19)

Perhaps a better positive name for this position is pragmatism. Putnam himself calls it verificationism in Putnam 1980. Whatever the name, Putnam maintains his indispensabilism. In fact, Putnam broadens his use of the indispensability argument, in his anti-realist phase, and maintains the link between empirical evidence and mathematical truth.

> [R]ejecting the spectator point of view, taking the agent point of view towards my own moral beliefs, and recognizing that *all* of the beliefs that I find indispensable in life must be treated by me as assertions which are true or false (and which I believe are true), without an invidious distinction between *noumena* and *phenomena*, is not the same thing as lapsing back into metaphysical realism about one's own moral beliefs any more than taking this attitude towards one's beliefs about commonsense material objects or towards causal beliefs or mathematical beliefs means lapsing back into metaphysical realism about commonsense

[11] Putnam had earlier claimed, pace Quine, that there are analytic truths, but they are not interesting, while maintaining his rejection of the a priori. See Putnam 1976: p 95 et. seq.

> objects, or causality, or mathematical objects/modalities. It also does not require us to give up our pluralism or our fallibilism: one does not have to believe in a unique *best* moral version, or a unique *best* causal version, or a unique *best* mathematical version; what we have are *better and worse* versions, and that *is* objectivity." (Putnam 1987: 77)

Putnam's anti-realism arises largely out of considerations from model theory, especially of the Löwenheim-Skolem theorem. The downward version of the theorem states that any formal theory whose theorems assert the existence of non-denumerably many objects, as any mathematical theory strong enough to include the real numbers will, will also have denumerable models. Any sufficiently strong theory, including ones which might count as a best theory for Quine and the realist Putnam, will have non-standard models. A theory can not determine an intended model.

Furthermore, Skolemite reinterpretations allow for interpretations which conflict, or appear to, with the intended interpretation. A term representing a non-denumerable set can be modeled by a countable set. Whether the term is countable or not, Putnam argues, is relative to the model. Truth values of sentences containing such terms are similarly relative.

Since our evidence for an empirical theory can not fix a model, we might seek non-empirical evidence, though doing so might violate naturalist constraints. Putnam rejects this tactic, again affirming indispensabilist tenets.

> [T]he theoretical constraints we have been speaking of must, on a naturalistic view, come from only two sources: they must come from something like human decision or convention, whatever the source of the 'naturalness' of the decisions or conventions must be, or from human experience, both experience with nature (*which is undoubtedly the source of our most basic 'mathematical intuitions'*, even if it be unfashionable to say so), and experience with 'doing mathematics'. (Putnam 1980: 5, emphasis added)

Putnam's anti-realism arises partly from his recognition that indispensabilism entails limits on mathematics. He argues that Gödel's Axiom of Constructibility, *V=L*, which says that every set is constructible, and which is not derivable from any common axiomatization of set theory, lacks a truth value. Empirical evidence, in the form of theoretical and operational constraints, does not settle the matter.

> [I]f the 'intended interpretation' is fixed only by theoretical plus operational constraints, then if 'V=L' does not follow from those theoretical constraints - if we do not *decide* to make V=L true or to make V=L false - then there will be 'intended' models in which V=L is *true*. If I am right, then the 'relativity of set-theoretic notions' extends to a *relativity of the truth value of 'V=L'* (and by similar arguments, of the axiom of choice and the continuum hypothesis as well). (Putnam 1980: 8)

Putnam's broader anti-realism is just an extension of this same argument, that we can not fix an intended model from within a theory, whether mathematical or scientific. Michael Levin has argued that Putnam's contention that reference for scientific theories becomes unfixed is based firmly on this unfixing of reference within mathematical theories, but that Putnam does not succeed in establishing the mathematical case. He criticizes Putnam for claiming that empirical evidence can tell us anything about the truth value of V=L, since that claim is about pure sets, about pure mathematics. "Empirical facts about the results of measurement cannot confirm or refute V=L because V=L is a statement about pure sets." (Levin 1997: 61)

Putnam's extension of his anti-realist conclusion to empirical theory is unmotivated, since we have external evidence in this case to nail down the references of our terms, and rule out unintended interpretations. The claim in mathematics is equally implausible, since the same empirical evidence is at issue. Again, Putnam has indispensability in the background.

Putnam himself implies a way to avoid anti-realist conclusions from the puzzles of model theory. "So to follow this line - which is, indeed, the right one, in my view - one needs to develop a theory on which interpretations are specified *other* than by specifying models." (Putnam 1980: 14)

One option, which I pursue for a moment in contrast to Putnam's indispensabilism, would be a traditional realism not based exclusively on the construction of axiomatic theories. Putnam limits the traditional realist's resources in such a way as to make this alternative unappealing. He compares two conflicting systems of set theory, while continuing to assume that no empirical evidence will settle the truth value of the axiom of choice. The first system includes the axiom of choice. The second he attributes to hypothetical extra-terrestrials who reject this axiom. Putnam says that we have no basis to discriminate between the two systems, portraying the realist as committed to arbitrary decisions. "But if both systems of set theory - ours and the extra-terrestrials' - count as *rational*, what sense does it make to call one *true* and the others *false*? From the Platonist's point of view there is no trouble in answering this question. 'The axiom of choice is true - true in *the* model, he will say (if he believes the axiom of choice)." (Putnam 1980: 10)

This is, admittedly, one possible response for the realist, though it is not the best one. There is mathematical evidence available beyond the empirical constraints. Even without further evidence, the realist can see both systems of set theory as independently true, of different universes of sets. The realist is not forced to make arbitrary and unjustified pronouncements on unanswerable questions.

Putnam attacks a straw man when he derides the realist for relying on mysterious faculties. "[T]here is the extreme Platonist position, which posits non-natural mental powers of directly 'grasping' forms (it is characteristic of this position that 'understanding' or 'grasping' is itself an irreducible and unexplicated notion)..." (Putnam 1980: 1)

It is not essential to realism that its key notions are inexplicable, even if they are so far not fully explained. The realist may explain mathematical understanding in terms of reliability, consistency, coherence, and intuition. These factors are not, with the possible exception of the last one, objectionable to the naturalist.

Indispensability, in the guise of restricting mathematical justification to empirical evidence, is a major source of Putnam's anti-realism in mathematics. Since Putnam's general anti-realist case depends on the mathematical case, indispensability is a major cause of Putnam's broader anti-realism.

The anti-realist Putnam relies on indispensability to establish internal realist truth for mathematics. He emphasizes naturalism and disparages non-indispensabilist realism. He subordinates mathematical practice to scientific practice. His concerns about the Axiom of Constructibility come from his recognition that empirical evidence is insufficient to establish truth values for a variety of mathematical claims. These are all indications of indispensability.

Here is another way to the same end: Putnam's arguments that no statement is unrevisable, and that there is no absolute a priori, all rest on his analysis of Euclidean geometry. If we do not accept his claim that relativity theory showed that something we thought a priori turned out not to be, then we have no reason to believe there is no absolute a priori. Putnam's anti-realism is unmotivated, unless he can defend his indispensability argument. In the next section, I show that he can not.

§7: Putnam's Success Argument

I believe that Quine's indispensability argument is ultimately unsuccessful, but my reasons for that are beyond the range of this paper. Here, I show that Putnam's indispensability argument, his success argument, does not succeed. It is, for the most part, an appeal to the practical utility of mathematics, and so is no argument for mathematical truth.

Putnam provides the following seed of his success argument. "I have argued that the hypothesis that classical mathematics is largely true accounts for the success of the physical applications of classical mathematics (given that the empirical premisses are largely approximately true and that the rules of logic preserve *truth*)." (Putnam 1975a: 75)

Putnam's success argument for mathematics is analogous to, and may be compared with, his success argument for scientific realism, which I discuss briefly and set aside. The scientific success argument relies on the claim that any position other than realism makes the success of science miraculous.

(SS) SS.1: Scientific theory is successful.
SS.2: There must be a reason for the success of science.
SS.3: No positions other than realism in science provide a reason.
SS.C: So, realism in science must be correct.

Given the relatively uncontroversial SS.1 and SS.2, the argument for realism in science rests on SS.3, and the miracles argument. But, strictly false theories such as Newtonian mechanics can be extremely useful and successful. If realism were the only interpretation which accounted for the success of science, then the utility of many false scientific theories is left unexplained. An instrumentalist interpretation on which theories may be useful without being true better accounts for the utility of false theories.

There are probably good responses to this quick criticism, but refuting SS is besides the point, here. My point here is merely that the miracles argument is best understood as an argument for scientific realism, and not for mathematical realism. I now set it aside and examine Putnam's analogous but independent success argument for mathematics.

(MS) MS.1: Mathematics succeeds as the language of science.
MS.2: There must be a reason for the success of mathematics as the language of science.
MS.3: No positions other than realism in mathematics provide a reason.
MS.C: So, realism in mathematics must be correct.

To see that MS is independent of SS, consider that even if science were interpreted instrumentally, mathematics may be justified by its applications. The problems with scientific realism may focus on the incompleteness and error of contemporary scientific theory. These problems need not infect our beliefs in the mathematics applied. A tool may work fine, even on a broken machine.

MS.1 is inoffensive even to the nominalist who thinks we can dispense with mathematics. MS.2 is just a demand for an account of the applicability of mathematics to scientific theory. MS, like SS, rests on its third step.

MS goes wrong in two ways. First, the third premise is weak. Second, even if one could establish that premise, and the argument, the mathematical realism it would establish would suffer Unfortunate Consequences.

I start with the second class of criticisms. The mathematical realism purportedly generated by any indispensability argument, one which satisfies all the Essential Characteristics, will suffer some Unfortunate Consequences.[12]

UC.1: Restriction: Our commitments are to only those mathematical objects required by empirical science. Mathematical results which are not applied in scientific theory are illegitimate. Mathematical objects not required for applied mathematics do not exist.
UC.2: Ontic Blur: The indispensabilist has no basis on which to make an abstract/ concrete distinction.
UC.3: Modal Uniformity: The existence of mathematical objects is as contingent as, and contingent on, the physical world.
UC.4: Temporality: Mathematical objects exist in time.
UC.5: Aposteriority: Mathematical objects are known a posteriori.
UC.6: Uniqueness: Any debate over the existence of a mathematical object will be resolved by the unique answer generated by empirical theory.

Putnam's success argument for mathematical realism retains all the Essential Characteristics of an indispensability argument. His work in all stages is professedly committed to EC.1, Naturalism. The argument commits to Theory Construction, EC.2, at MS.2. Putnam sees mathematical realism as arising from the construction of that best science, which is EC.3, Mathematization. And Putnam defends the primacy of scientific practice, EC.4, mainly to avoid a mystical platonism. Since MS has all the Essential Characteristics, it is burdened with the resulting Unfortunate Consequences.[13]

But the Unfortunate Consequences are really moot, since MS.3 is false. Putnam's argument for it is essentially a rejection of the argument that mathematics could be indispensable, yet not true. "It is silly to agree that a reason for believing that p warrants accepting p in all scientific circumstances, and then to add 'but even so it is not *good enough*'."

[12] I develop these in greater detail elsewhere.

[13] Restriction is even more of a problem for Putnam than it is for Quine, since the holist can argue for slightly more mathematics on the basis of simplicity and rounding-out of theory.

(Putnam 1971: 356)

For the holist, Putnam's argument has some force. For, the holist has no external perspective from which to evaluate the mathematics in scientific theory as instrumental. Adrift on Neurath's ship, he can not say, "Well, I commit to mathematical objects within scientific theory, but I don't really mean that they exist."

For Putnam, who rejects holism, instrumentalist interpretations of the mathematics used in scientific theory are more compelling. For, he is no longer constrained to limit existence claims to the quantifications of our best theory. He is free to adopt an eleatic principle, for example, as the fundamental criterion for existence. The eleatic, of course, rejects mathematical objects.

More importantly, we need only one account of the applicability of mathematics to the empirical world other than the indispensabilist's to refute MS.3. Mark Balaguer's plenitudinous platonism, for example, suffices, since it claims that mathematics provides a theoretical apparatus which applies to all possible states of the world.[14]

One could amend MS.3:

MS3*: Realism best explains the success of mathematics as the language of science.

This change does not help, though, since realism does not best explain the application of mathematics. Realism is just the claim that some mathematical claims are non-vacuously true. It says nothing about the applicability of mathematics to the physical world. Moreover, dispensabilist constructions like that of Field 1980 erode confidence in MS.3 by presenting an

[14]See Balaguer 1998, Chapters 3 and 4.

alternate account of why mathematics is useful in science.

My rejection of MS contained two distinct elements. First, there are other, and better, accounts of the application of mathematics to physical theory. Any application which actually explains the connection between abstract mathematical objects and the physical world will be preferable to Putnam's, which takes this relationship as brute. Second, even if we were to accept the validity of MS, the mathematics yielded would still suffer the Unfortunate Consequences.

These elements, combined, reveal a tension in MS. The objects justified by indispensability are concrete, known a posteriori, and exist contingently and temporally. So, indispensability can not establish mathematical knowledge. But, if it could, the account of why mathematics is useful in science would clearly be missing since the mathematical objects inhabit a separate, abstract realm.

§8: The Failure of Putnam's Unifying Themes

In the introduction to *Mathematics, Matter and Method*, Putnam writes that the unity of his work to that point consists of four doctrines:

PD.1: Mathematical and scientific realism;
PD.2: Rejection of the absolute a priori;
PD.3: Rejection of the link between the factual and the empirical; and
PD.4: That mathematics is both empirical and quasi-empirical.

None of the themes Putnam cites are successful. He arrives at PD.4 by conflating empiricism with fallibilism. His arguments only establish that mathematical claims are defeasible, not that they are empirical. Once he adopts quasi-empiricism, though, he allows real empirical justification of mathematical claims.

Putnam takes the indispensability argument to support PD.3, which is his “[R]ejection of the idea that ‘factual’ statements are all and at all times ‘empirical’, i.e. subject to experimental or observational test.” (Putnam 1974: vii) Mathematical statements are supposed to be factual, but not empirical. Putnam fails to see that the empirical justification for mathematics at the core of his indispensability argument makes mathematics empirical, i.e. subject to experimental or observational test. Since mathematics is empirical, on Putnam’s account, the link remains.

His claim to PD.1, including realism about mathematical objects and necessity, is a stretch. The volume which contains this claim includes papers which promote deductivism and modalism in addition to realism. Certainly his later work is a rejection of realism, even if he calls it ‘internal realism’. Putnam’s success argument, his version of the indispensability argument and his clearest defense of mathematical realism, also failed.

As for PD.2, the version of apriorism which Putnam rejects entails necessity and indefeasibility. Putnam is right that anyone who held this position must have been mistaken. It is plausible to read figures in the history of philosophy, like Descartes, as holding it. Kant’s view of Euclidean geometry as the necessary, indefeasible structure of space was surely a mistake. Putnam takes the wrong lesson from this. He makes the a priori relative instead of fallible.

The failures of PD.1 - PD.4 aside, indispensability is a deeper unifying theme. Deductivism and modalism only commit to limited elements of the indispensability thesis, since those positions do not commit to the existence of mathematical objects. In terms of the Essential Characteristics, Putnam’s deductivism contains EC.1, EC.2, and EC.4, though it restricts the claim of EC.3, Mathematization. Putnam’s modalism has the same characteristics, though in place of EC.3, he recognizes what one might call Modalization, that empirical science is committed to possibilia. Putnam's realism contains all the Essential Characteristics, as does his anti-realism, though he moderates their ramifications.

In the despairing, Putnam 1994, "Philosophy of Mathematics: Why Nothing Works", he criticizes both intuitionism and formalism for conflicting with indispensabilist tenets. The intuitionist, Putnam argues, can not connect his mathematical logic with empirical logic. He changes the meaning of the logical connectives. Implication, in intuitionist mathematics, means that there is a constructive proof procedure in which the consequent follows from the antecedent. "While the assumption that there are such things as verifications ('proofs') of isolated statements may be all right in mathematics, it is not in physics, as many authors have pointed out. So what does $\supset$ mean in an *empirical statement*?" (Putnam 1994: 509)

He assimilates formalism with a broader program, and again criticizes the program for failing to adhere to indispensabilist principles. "In short, the formalist seems to be really a kind of philosophical nominalist - and nominalism is (it is generally believed) inadequate for the analysis of empirical discourse." (Putnam 1994: 502)

Putnam, despite surface adjustments, despite flirtations with deductivism and modalism, and throughout his attempts to make sense of quasi-empiricism, or internal realism, always held the basic thesis of mathematical empiricism. "It is only when the language of mathematics is considered as an integral part of the language of science as a whole - in other words, considered in its relation to *empirical* science - that the reason for making these [mathematical] assumptions can become clear." (Putnam 1956: 87)

Bibliography

Balaguer, Mark. 1998. *Platonism and Anti-Platonism in Mathematics*. New York: Oxford University Press.

Burgess, John, and Gideon Rosen. 1997. *A Subject with No Object*. New York: Oxford.

Field, Hartry. 1988. "Realism, Mathematics and Modality." Reprinted in Field, *Realism, Mathematics, and Modality*. Oxford: Basil Blackwell.

Field, Hartry. 1980. *Science Without Numbers*. Princeton: Princeton University Press.

Levin, Michael. 1997. "Putnam on Reference and Constructible Sets." *British Journal of the Philosophy of Science* 48: 55-67.

Putnam, Hilary. 1994. "Philosophy of Mathematics: Why Nothing Works." In *Words and Life.* Cambridge: Harvard University Press.

Putnam, Hilary. 1987. *The Many Faces of Realism*. LaSalle: The Open Court Press.

Putnam, Hilary. 1983. *Realism and Reason*: Philosophical Papers, Vol. III. Cambridge: Cambridge University Press.

Putnam, Hilary. 1981. *Reason, Truth and History*. Cambridge: Cambridge University Press.

Putnam, Hilary. 1980. "Models and Reality." Reprinted in Putnam 1983.

Putnam, Hilary. 1979. "Analyticity and Apriority: Beyond Wittgenstein and Quine." Reprinted in Putnam 1983.

Putnam, Hilary. 1978. "There Is At Least One A Priori Truth." Reprinted in Putnam 1983.

Putnam, Hilary. 1976. "'Two Dogmas' Revisited." Reprinted in Putnam 1983.

Putnam, Hilary. 1975c. *Mind, Language, and Reality: Philosophical Papers, Vol. II.* Cambridge: Cambridge University Press.

Putnam, Hilary. 1975b. *Mathematics, Matter, and Method: Philosophical Papers, Vol. I.* Cambridge: Cambridge University Press.

Putnam, Hilary. 1975a. "What is Mathematical Truth?" Reprinted in Putnam 1975b.

Putnam, Hilary. 1974. "Science as Approximation to Truth." Reprinted in Putnam 1975b.

Putnam, Hilary. 1971. *Philosophy of Logic*. Reprinted in Putnam 1975b.

Putnam, Hilary. 1968. "The Logic of Quantum Mechanics." Reprinted in Putnam 1975b.

Putnam, Hilary. 1967b. "Mathematics Without Foundations." Reprinted in Putnam 1975b.

Putnam, Hilary. 1967a. "The Thesis that Mathematics is Logic." Reprinted in Putnam 1975b.

Putnam, Hilary. 1962. "The Analytic and the Synthetic." Reprinted in Putnam 1975c.

Putnam, Hilary. 1957. "Three-Valued Logic." Reprinted in Putnam 1975b.

Putnam, Hilary. 1956. "Mathematics and the Existence of Abstract Entities." *Philosophical Studies* 7: 81-88.

Quine, W.V. 1980. *From a Logical Point of View*. Cambridge: Harvard University Press.

Quine, W.V. 1978. "Success and the Limits of Mathematization." In *Theories and Things*. Cambridge: Harvard University Press.

Quine, W.V. 1969. "Existence and Quantification." In *Ontological Relativity and Other Essays*. New York: Columbia University Press.

Quine, W.V. 1960. *Word & Object*. Cambridge: The MIT Press.

Quine, W.V. 1951. "Two Dogmas of Empiricism." Reprinted in Quine 1980.

Quine, W.V. 1948. "On What There Is." Reprinted in Quine 1980.

Resnik, Michael. 1995. "Scientific vs. Mathematical Realism: The Indispensability Argument. *Philosophia Mathematica (3)* 3: 166-174.

.

Wigner, Eugene P. 1959. "The Unreasonable Effectiveness of Mathematics in the Natural Sciences." In, *Symmetries and Reflections: Scientific Essays of Eugene P. Wigner*. Bloomington: Indiana University Press.