Why the Indispensability Argument Does Not Justify Belief in Mathematical Objects

Russell Marcus, Ph.D.

Chauncey Truax Post-Doctoral Fellow

Department of Philosophy, Hamilton College

198 College Hill Road

Clinton NY 13323

rmarcus1@hamilton.edu

(315) 859-4056 (office)

(315) 381-3125 (home)

(917) 903-7483 (mobile)

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Abstract:

The Quine-Putnam indispensability argument concludes that we should believe that mathematical objects exist because of their ineliminable use in scientific theory. I argue that none of the objects in which the indispensability argument justifies belief are mathematical objects. I first present a traditional characterization of mathematical objects. I then propose a general formulation of the indispensability argument. I argue that the objects to which the indispensability argument refers do not have six important properties traditionally ascribed to mathematical objects. One can not justify a platonist mathematical ontology with an empiricist epistemology.

§1: Mathematical Objects

The Quine-Putnam indispensability argument may be used to try to justify beliefs in the existence of mathematical objects on the basis of their ineliminable use in science. The indispensabilist claims that we can eschew traditional appeals to mathematical intuition, or other platonist epistemology, yet maintain a substantial, abstract mathematical ontology.[1] The argument is most often criticized for its claim that mathematical objects are ineliminable from scientific theory. In this paper, I argue that the so-called mathematical objects to which the indispensability argument refers, the objects to which scientific theory may be taken to be committed, are not really mathematical objects. Thus, the indispensability argument does not justify beliefs in mathematical objects.

I will discuss the indispensability argument shortly, but it will help to start with a characterization of mathematical objects. For simplicity, the only mathematical objects I shall consider will be sets, due to common, though not universal, presumptions about the reducibility of all mathematical objects to sets.[2] The characterization of sets that I will provide is traditional. Readers may find the traditional conception contentious. My point is that the indispensability argument does not justify beliefs in mathematical objects, as traditionally conceived.

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[1] Aside from Quine and Putnam's work, see Resnik 1993: "A Naturalized Epistemology for a Platonist Mathematical Ontology."

[2] Nothing I say in the paper depends on the reductive presumption; all results could be generalized. Critics of the reductive presumption include structuralists motivated by Benacerraf 1965, which denies a unique reduction. In another direction, category theorists may hold that reduction should be instead to more fundamental categories.

I take it that sets are abstract objects, lacking any spatio-temporal location.[3] The universe of sets is described by various standard axiomatizations; where different axiomatizations conflict we find disagreement about the nature and extent of the set-theoretic universe.[4] Their existence is not contingent on our existence, nor is it contingent on the existence of any physical objects.[5] Furthermore, I take it that mathematics is a discipline autonomous from empirical science; mathematical standards are independent of application.[6] Mathematical methodology is a priori.[7]

Each of the properties of sets that I have mentioned has been denied of mathematical objects, just as the existence of mathematical objects has been denied. Still, these characteristics constitute, at least in part, the standard starting point for discussions of the nature of mathematics and mathematical objects.[8] In this paper, I remain agnostic on whether mathematical objects exist. I also remain agnostic on whether sets must have all of the above characteristics. My claim is merely that any objects which lack all of the above characteristics should not be called

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[3] Attempts to locate mathematical objects with their concrete members lead to substantial difficulties, as Frege argued against Mill (see Frege 1980, §7-§9), and as Mark Balaguer argues against Penelope Maddy (see Balaguer 1994).

[4] Or universes. Balaguer 1998 argues that any consistent axiomatization truly describes a universe of sets, even if it conflicts with other consistent axiomatizations.

[5] I speak here of pure sets. Sets with ur-elements may exist contingently.

[6] We do sometimes pursue mathematics in order to solve specific problems in empirical science. My claim is that we have criteria for determining whether to accept a mathematical assertion which are independent from the application of that assertion to empirical science.

[7] There are empirical aspects to mathematical methods, of course: knowledge of who proved which theorems, say, and observations of inscriptions. But, such empirical claims do not suffice for mathematical justification.

[8] They are consistent, for example, with what James Robert Brown calls the "mathematical image" (Brown 1999: 1-7) and with Stuart Shapiro's traditional picture (Shapiro 2000: 21-23).

mathematical objects. And, the so-called mathematical objects which the indispensability argument says we should believe exist lack all of the above characteristics.

§2: Indispensability Arguments

An indispensability argument is an inference to the best explanation which transfers evidence for one set of claims to another. For example, we believe that atoms exist, despite having no direct perception of them, because they play an indispensable role in atomic theory, which we believe. The Quine-Putnam indispensability argument concludes that we should believe that mathematical objects exist since our best scientific theories ineliminably refer to them.[9] In this section, I present distinct versions of the argument, which I attribute to Quine and to Putnam. In §3, I propose a general characterization of these arguments. In §4, I discuss various unfortunate consequences of any instance of the argument.

Though Quine alluded to an indispensability argument in many places, he never presented a detailed formulation.[10] The method underlying the argument involves gathering our physical laws and casting them in a canonical language of first-order logic. The commitments of this regimented theory may be found by examining its quantifications.

(QI) QI.1: We should believe the (single, holistic) theory which best accounts for our sense experience.
QI.2: If we believe a theory, we must believe in its ontic commitments.
QI.3: The ontic commitments of any theory are the objects over which that theory first-order quantifies.

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[9] Field 1980 and its legacy remain the focus of the debate over whether such reference is eliminable. Burgess and Rosen 1997 collects a variety of strategies for removing mathematical elements from scientific theory.

[10] For a selection of such allusions, see Quines 1939, 1948, 1951, 1955, 1958, 1960, 1978, and 1986.

QI.4: The theory which best accounts for our sense experience first-order quantifies over mathematical objects.
QI.C: We should believe that mathematical objects exist.

While it is obvious that scientists use mathematics in developing their theories, it is not obvious, prima facie, why the uses of mathematics in science should force us to believe in the existence of abstract objects. When we study the interactions of charged particles, we rely on Coulomb's Law, which states that the electromagnetic force between two charged particles is proportional to the charges on the particles and, inversely, to the distance between them.

(CL) $F = k\ |q_1 q_2| / r^2$
where the electrostatic constant $k \approx 9 \times 10^9\ Nm^2/c^2$

CL refers to a real number, k, and employs mathematical functions like multiplication and absolute value. Still, we use Coulomb's Law to study charged particles, not to study mathematical objects, which have no effect on those particles.

The plausibility of Quine's indispensability argument thus depends on both his holism, at QI.1, and his method for determining ontic commitment, at QI.3. I believe that Quine's method does not accurately yield our commitments. In fact, I believe that QI.1, QI.2, and QI.4 are false, and that QI.3 only avoids falsity as a fruitless definition of ontic commitment.[11] But, my goal in

[11] Briefly: Against QI.1, there are reasons to believe that our best theory is not, in whole, true; see Cartwright 1983 and van Fraassen 1980. Further, it seems that our best theory is not (confirmation) holistic, at least regarding mathematical objects; see Sober 1993. Against QI.2, good theories may make posits which are best interpreted instrumentally; see Azzouni 2004. Against QI.3-4, first-order logic provides an insufficient framework for mathematics and physics; see Shapiro 1991.

this paper is to grant the validity of QI, and show that QI.4 and QI.C are false because of their use of 'mathematical objects'.

Variations on Quine's indispensability argument are available, some of which eschew Quine's holism and method for determining ontic commitment. Putnam at one point defended Quine's argument, but also presented an independent indispensability argument, the success argument.[12]

(PS) PS.1: Mathematics succeeds as the language of science.
PS.2: There must be a reason for the success of mathematics as the language of science.
PS.3: No positions other than realism in mathematics provide a reason.
PS.C: So, realism in mathematics must be correct.[13]

PS does not rely on Quine's method for determining ontic commitment. PS also avoids Quine's confirmation holism which spreads evidential support throughout all elements of our best theory, including the mathematical ones. PS relies only on the pragmatic issue of mathematical success in science.

Other versions of the argument are available. Colyvan 2001 defends a Quinean argument.[14] Resnik 1997 presents both a Quinean argument and a pragmatic argument, related to PS, which appeals to casual uses of mathematics in science to support the structuralist's

[12] For Putnam's Quinean argument, see Putnam 1956 and Putnam 1971. The distinct success argument is also found in Putnam 1971, as well as in Putnam 1975 and Putnam 1994.

[13] Putnam talks about mathematical realism in lieu of ontic commitments to mathematical objects, but the same point is at issue.

[14] Colyvan's presentation of the Quinean argument does not make clear the argument's reliance on Quine's method for determining ontic commitments. He thus confounds QI with PS.

commitment to mathematical objects. Maddy 1992 attempts to extend the Quinean argument. I proceed to provide a general characterization of the argument which covers all of these, and other, indispensability arguments.

§3: The Essential Characteristics of Indispensability Arguments

The following are essential characteristics of any indispensability argument which concludes that we should believe that mathematical objects exist.

EC.1: Naturalism: The job of the philosopher, as of the scientist, is exclusively to understand our sensible experience of the physical world.
EC.2: Theory Construction: In order to explain our sensible experience we construct a theory, or theories, of the physical world. We find our commitments exclusively in our best theory or theories.
EC.3: Mathematization: Some mathematical objects are ineliminable from our best theory or theories.
EC.4: Subordination of Practice: Mathematical practice depends for its legitimacy on empirical scientific practice.

It follows from Naturalism that we never need to explain mathematical phenomena, like the existence of oddly many twin primes, for their own sake. Ultimately, the justification of any mathematical belief must be grounded in our sense experience.

Theory Construction tells us where to look for ontic commitments, but does not settle a particular procedure for determining them. If we drop Quine's method, we need another one. Shapiro 1993, for example, urges that we adopt a structuralist criterion. Some indispensabilists leave their methods for determining ontic commitments obscure, or implicitly rely on Quine's criterion. Theory Construction rules out independent, non-empirical justification of mathematical claims.[15]

Mathematization is an empirical claim about the needs of theory construction. I call this claim empirical since it seems to be an empirical question whether we can formulate nominalist alternatives to all good scientific theories, including future theories. Again, my concern in this paper is to deny Mathematization.

Subordination of Practice, implicit in the other characteristics, emphasizes the relationship between mathematics and empirical science for the indispensabilist. Rejecting Subordination of Practice would entail either adopting an alternate justification for our mathematical beliefs, or denying that mathematical practice yields any commitments.[16]

§4: The Unfortunate Consequences, and Their Links to the Essential Characteristics

The essential characteristics entail some unfortunate consequences for the mathematical objects to which the indispensability argument refers. Some of these unfortunate consequences have already been noticed. This paper serves to compile these complaints, and extend and explain their scope.

First, since Mathematization and Naturalism rule out any alternate justifications for mathematical claims, the indispensabilist has no commitments to mathematical objects which are

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[15] One might think that an indispensabilist may also admit a strictly mathematical epistemology. If so, there would seem to be two routes to justification of mathematical beliefs, one independent of empirical science, the other relying on it. But, the former route would do all the real work. Science might explain why we have our beliefs, but it would not justify them.

[16] Maddy 1992 denies Subordination of Practice, abandoning the indispensability argument in favor of justification on the basis of mathematical practice.

not required for empirical science.[17] Call this consequence Restriction. It is difficult to say precisely which mathematical objects the indispensability argument would justify, i.e. how much mathematics empirical science actually needs. Burgess and Rosen suggest that there is historical consensus that science needs no more than analysis.[18] Feferman 1998 argues that predicative set theory will suffice. The point at which the indispensabilist draws the line between justified and unjustified mathematical beliefs is unimportant here. What is relevant is the existence of a division, one which Quine embraces. "I recognize indenumerable infinities only because they are forced on me by the simplest known systematizations of more welcome matters. Magnitudes in excess of such demands, e.g., $\beth_{\omega}$ or inaccessible numbers, I look upon only as mathematical recreation and without ontological rights"(Quine 1986: 400).

There are really three problems of Restriction. First, the existence of a division with no mathematical basis between justified and unjustified mathematical objects is itself counter-intuitive. Second, the restrictions on the indispensabilist do not merely apply to the outer regions of set theory. Justifications of mathematical claims vary with shifts in our best scientific theory. As science progresses, and uses new mathematical tools, the mathematics which is justified can grow, and these changes can occur though no mathematical progress need be made. Third, such changes can, in principle, decrease the scope of legitimate mathematics. Maddy 1992 suggests that all of science could, in principle, become quantized. In such circumstances, we could lose

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[17] Supplementing the indispensability argument to justify unapplied results, as Maddy 1992 does, or appealing to a priori intuition or logicism, renders the original argument superfluous. If logicism, intuition, or mathematical practice can justify belief in large cardinals not used in empirical science, then surely it can justify beliefs in sets which are applied.

[18] See Burgess and Rosen 1997: 76.

justifications for belief in the real numbers.

A second unfortunate consequence, call it Ontic Blur, arises directly from Theory Construction, which entails that the indispensabilist can not differentiate between abstract and concrete objects. The indispensabilist's theory is constructed to explain or represent phenomena involving ordinary objects. "Bodies are assumed, yes; they are the things, first and foremost. Beyond them there is a succession of dwindling analogies" (Quine 1981: 9).

As these analogies dwindle, the abstract/concrete distinction blurs. Indeed, the terms 'abstract' and 'concrete' become rather meaningless for the holist, vulgar terms in which the learned may only lightly indulge. As Parsons notes,

> Although Quine makes some use of very general divisions among objects, such as between 'abstract' and 'concrete', these divisions do not amount to any division of *senses* either of the quantifier or the word 'object'; the latter sort of division would indeed call for a many-sorted quantificational logic rather than the standard one. Moreover, Quine does not distinguish between objects and any more general or different category of 'entities' (such as Frege's *functions*). (Parsons 1983: 377)

Furthermore, Quine himself wonders if such distinctions are sustainable.

> [O]dd findings [in quantum mechanics] suggest that the notion of a particle was only a rough conceptual aid, and that nature is better conceived as a distribution of local states over space-time. The points of space-time may be taken as quadruples of numbers, relative to some system of coordinates... We are down to an ontology of pure sets. The state functors remain as irreducibly physical vocabulary, but their arguments and values are pure sets. The ontological contrast between mathematics and nature lapses. (Quine 1986: 402)[19]

For Quine, the abstract/concrete distinction must be made within science. But scientific

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[19] See also Quine 1978; Quine 1960: 234; Quine 1974: 88; and Quine 1969: 98.

theory does not support the distinction. The quantifier univocally imputes existence. All commitments are made as values of bound variables. We must classify the indispensabilist's purported abstract objects with the concrete objects they are used to explain or describe.

Independently of the indispensability argument, we can establish a criterion for abstractness, e.g. on the basis of what Balaguer calls the principle of causal isolation (PCI) of mathematical from empirical objects. With an epistemology for mathematics separate from that for empirical science, the claim that mathematical objects are abstract is plausible. But, PCI is off limits to the indispensabilist. In fact, ontic blur, the rejection of PCI, is definitive of the indispensability argument, as Balaguer notes. "The Quine-Putnam argument should be construed as an argument not for platonism or the truth of mathematics but, rather, for the falsity of PCI" (Balaguer 1998: 110).

When we combine Theory Construction with Mathematization, we find that the indispensabilist's mathematical objects do not exist necessarily. Mathematical objects are posited to account for our experience of a world which exists contingently. If the world were different, if it contained different physical laws, then scientific theory could require different objects. Call this consequence Modal Uniformity.[20]

To illustrate Modal Uniformity, suppose that charge is actually a continuous property of real particles in this world. The indispensabilist thus alleges that the world contains continuous functions. Further, suppose that in a different possible world, there are no continuous properties. In that world, says the indispensabilist, there are no continuous functions. Whether there are

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[20] Colyvan 2000 embraces the contingent existence of mathematical objects for the indispensabilist. That paper is in part a response to discussion among Crispin Wright, Bob Hale, and Hartry Field about Field's claim that mathematical objects contingently do not exist.

continuous functions depends on whether the world contains continuous properties.

All hope for modality may not be lost for the indispensabilist. For, there are several notions of necessity. When one asserts that the world is possibly Newtonian, even if relativistic, one may refer to a weak notion of physical necessity on which phenomena in accord with scientific laws follow from them necessarily. More strongly, a statement may be logically necessary, which may be construed as entailing a contradiction when negated, or as being either a logical law or following from one. Even more strongly, a statement may be metaphysically necessary, or true in all possible worlds. Kripke alleges that some identity statements, ones flanked by rigid designators, like the identity of water and H_2O, are metaphysically necessary. Some naturalists claim insight into Kripkean metaphysical necessities. Perhaps the indispensabilist could claim, similarly, that some mathematical claims are necessary.

There are two reasons to be skeptical about an indispensabilist's attempt to avoid modal uniformity by appeal to Kripkean metaphysical necessities. First, the indispensabilist's naturalism would seem to debar such claims. Explanations of Kripkean metaphysical necessities tend to rely on a non-naturalist a priori intuition unavailable to the indispensabilist. Second, even if the indispensabilist could establish that some mathematical identity statements are metaphysically necessary, it would not follow that mathematical objects exist necessarily, or that we should believe that they do.

By linking the justifications for our beliefs in mathematics to the physical world, the indispensabilist may retain a weaker modality, like physical necessity, for mathematical claims. Unfortunately, the weaker notion is not the one traditionally imputed to mathematics, and is unsatisfactory. It would follow that under a different set of physical laws, two and two might not equal the square root of sixteen. While this idea may be alluring to some, it seems absurd. Only a stronger necessity will do justice to our intuitions that mathematical truths are broader than physical ones.

As a corollary of Modal Uniformity, mathematical objects are temporal, as well as contingent. For, if mathematical objects exist contingently, then there can be a time when they do not exist. If the existence of continuous functions depends on the existence of continuous physical quantities, then if the physical quantities were to be extinguished, the mathematical functions would disappear as well, and would have to be removed from our list of commitments. Again, mathematical objects are traditionally taken to be atemporal. "It would betray a confusion to ask, 'When did (or when will) these primes exist? At what time may they be found?'" (Burgess and Rosen 1997: 21).

While it is traditional to ascribe to mathematics an a priori methodology, the indispensabilist only provides an epistemology for empirical science. This single epistemology also entails that the indispensabilist's mathematical objects are, like concrete objects, known a posteriori. Indeed, many indispensabilists, like Quine, are motivated by a desire to avoid a priori epistemology.

Lastly, Subordination of Practice entails that any mathematical debate, like that over the Axiom of Choice, should be resolved not on mathematical terms, but on the basis of the needs of science. Chihara criticizes this indispensabilist subordination of mathematical practice. "It is suggested [by Quine] that which mathematical theory we should take to be true should be determined empirically by assessing the relative scientific benefits that would accrue to science from incorporating the mathematical theories in question into scientific theory. It is as if the mathematician should ask the physicist which set theory is the true one!" (Chihara 1990: 15).

For another example, consider the introduction, by Cardan, of complex numbers as solutions to quadratic equations with missing real roots. So-called imaginary, or impossible, numbers were derided, despite their mathematical uses.[21] Complex numbers simplified mathematics, since ad hoc explanations about why certain quadratic equations had two roots, others just one, and others none, were avoided. A fruitful field of study was born with geometric, graphical representations. The theory of complex numbers was not found to contain any inconsistency, aside from the conflict with a presupposition that all numbers were real numbers. Physical applications were later discovered, for example in representing inductance and capacitance as the real and imaginary parts of one complex number, instead of as two distinct reals.

For the mathematician, the legitimacy of complex numbers came early. The indispensabilist, prior to the discovery of their applicability, could not accommodate them. Even the analogy with negative numbers, which arose from similar disrepute, serves as no argument for the indispensabilist.[22] Lacking application, work with complex numbers was just mathematical recreation.

The indispensabilist will describe the discovery of an application for any mathematical objects as one of an empirical confirmation of its existence. Not all mathematical objects will be as lucky as the complex numbers. Consider two conflicting mathematical theories, like ZF + CH

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[21] See Kline 1972: 253-4; also 594-7, where Kline cites Euler calling complex numbers impossible.

[22] For an account of the disrepute of negative numbers, again see Kline 1972: 252-3.

and ZF + not-CH.[23] It is possible, in this case and others like it, that neither theory will admit of application/confirmation. It seems safe to presume that there will not be application for all of the transfinitely many, presumably consistent axiomatizations which result from adding axioms asserting different sizes of the continuum to ZF. There is no mathematical reason not to multiply set-theoretic universes. Perhaps there are multiple set-theoretic hierarchies; in some the continuum hypothesis holds, while in others it fails, and in different ways. The indispensabilist, committed to austerity in abstracta, adopts a mathematical theory only when it has physical application.

The problems of Subordination of Practice are even worse for theories which seem to suffer from empirical disconfirmation. Subordination of Practice entails that, in Resnik's terms, the indispensabilist's appeal to Euclidean rescues is limited.[24] When relativity supplanted classical mechanics, flat Euclidean space-time geometry was replaced by a curved, hyperbolic space-time. A Euclidean rescue defends the legitimacy of both the flat and curved geometries despite the change in physical theory. On a Euclidean rescue, all three possibilities concerning the parallel postulate are taken as axioms of consistent, unfalsified theories.

We can perform a Euclidean rescue any time a mathematical theory loses application in science. In such cases, the indispensabilist generally rejects the now-unapplied mathematics. The traditional response is the Euclidean rescue, unless the mathematics is shown inconsistent.

We can be sure that mathematicians working today in the farthest reaches of pure set

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[23] That is, Zermelo-Frankel set theory with the continuum hypothesis, or with its negation.

[24] See Resnik 1997: Chapter 7, §4.

theory do so without knowing that their work has any physical application. One may arise, or their work may never find use in empirical science. If the only justification for mathematics is in its application to scientific theory, then unapplied results are unjustified, even if they may eventually be useful. The indispensabilist makes the mathematician dependent on the scientist for the justification of his or her work.

The following are thus unfortunate consequences of the indispensability argument:

UC.1: Restriction: The indispensabilist's commitments are to only those mathematical objects required by empirical science.
UC.2: Ontic Blur: The indispensabilist's mathematical objects are concrete.
UC.3: Modal Uniformity: The indispensabilist's mathematical objects do not exist necessarily.
UC.4: Temporality: The indispensabilist's mathematical objects exist in time.
UC.5: Aposteriority: The indispensabilist's mathematical objects are known a posteriori.
UC.6: Methodological Subservience: Any debate over the existence of a mathematical object will be resolved, for the indispensabilist, by the needs of empirical theory.

§5: The Indispensability Argument Does Not Generate Mathematical Objects

Given the unfortunate consequences together with the traditional characterization of mathematical objects I listed in §1, it is clear that the indispensability argument can not justify beliefs in mathematical objects, even if our best scientific theory includes mathematical axioms. The indispensabilist's so-called mathematical objects retain none of their traditional characteristics. Still, the indispensabilist asserts that any regimentation of physics will require set-theoretic axioms in order to provide the required functions. So, what are the objects which satisfy these axioms, if not mathematical objects?

I would like to emphasize that the inclusion of mathematical axioms in a theory is no indication that the theory is committed to mathematical objects. The problem is not merely that theories do not determine their own models. The problem is that lots of objects can serve as models of mathematical axioms. Appropriately arranged peas can serve as models of finite portions of number theory. Field 1980 proposes using space-time regions to model geometry, including an axiom of continuity. Of course, peas will not suffice for ZF, but some less tractable and more plenitudinous objects can.

Quine urges a doctrine of gradualism from observables like trees, through subvisible objects like electrons, to space-time points and sets. The central point of this paper is that just as space-time points are not mathematical objects, neither are the indispensabilist's sets. They are ordinary empirical posits, somewhat less tractable, but no different in kind, than trees.

I hesitate to coin a name for these concrete, contingent, temporal, known-a-posteriori denizens of the indispensabilist's restricted universe. Still, a name will help us distinguish them from real sets. 'Quasi-sets' is taken. I propose 'indisets'. Indisets are concrete and temporal empirical posits, which unlike space-time points lack spatial location. As I mentioned in §1, there are questions whether the indispensability argument can justify belief even in indisets. But, the indispensability argument surely does not justify belief in mathematical objects.

References

Azzouni, Jody. 2004. *Deflating Existential Consequence: A Case for Nominalism*. Oxford University Press.

Azzouni, Jody. 1998. "On 'On What There Is'." *Pacific Philosophical Quarterly* 79: 1-18.

Azzouni, Jody. 1997. "Applied Mathematics, Existential Commitment, and the Quine-Putnam Indispensability Thesis." *Philosophia Mathematica (3)* 5: 193-209.

Balaguer, Mark. 1998. *Platonism and Anti-Platonism in Mathematics*. New York: Oxford University Press.

Balaguer, Mark. 1994. "Against (Maddian) Naturalized Platonism." *Philosophia Mathematica (3)*, 2: 97-108.

Benacerraf, Paul. 1965. "What Numbers Could Not Be." In Paul Benacerraf and Hilary Putnam, eds. *Philosophy of Mathematics: Selected Readings*, second edition. Cambridge: Cambridge University Press, 1983.

Brown, James Robert. 1999. *Philosophy of Mathematics: An Introduction to the World of Proofs and Pictures*. London: Routledge.

Burgess, John, and Gideon Rosen. 1997. *A Subject with No Object*. New York: Oxford.

Cartwright, Nancy. 1983. *How the Laws of Physics Lie*. Oxford: Clarendon Press.

Chihara, Charles. 1990. *Constructabiliy and Mathematical Existence.* Oxford: Oxford University Press.

Colyvan, Mark. 2001. *The Indispensability of Mathematics*. Oxford University Press.

Colyvan, Mark. 2000. "Conceptual Contingency and Abstract Existence." *The Philosophical Quarterly* 50: 87-91.

Feferman, Solomon. 1998. *In the Light of Logic*. New York: Oxford University Press.

Field, Hartry. 1980. *Science Without Numbers*. Princeton: Princeton University Press.

Frege, Gottlob. 1953. *The Foundations of Arithmetic*. Evanston: Northwestern University Press.

Kline, Morris. 1972. *Mathematical Thought from Ancient to Modern Times*. New York: Oxford University Press.

Kripke, Saul. 1980. *Naming and Necessity*. Cambridge: Harvard University Press.

Maddy, Penelope. 1992. "Indispensability and Practice." *The Journal of Philosophy* 89: 275-289.

Parsons, Charles. 1983. "Quine on the Philosophy of Mathematics." In *The Philosophy of W.V. Quine*, Lewis Edwin Hahn and Paul Arthur Schilpp, eds. La Salle: Open Court, 1986.

Putnam, Hilary. 1994. "Philosophy of Mathematics: Why Nothing Works." In *Words and Life.* Cambridge: Harvard University Press.

Putnam, Hilary. 1975. "What is Mathematical Truth?" Reprinted in Putnam, *Mathematics, Matter, and Method: Philosophical Papers, Vol. I.* Cambridge: Cambridge University Press, 1975.

Putnam, Hilary. 1971. *Philosophy of Logic*. Reprinted in Putnam, *Mathematics, Matter, and Method: Philosophical Papers, Vol. I.* Cambridge: Cambridge University Press, 1975.

Putnam, Hilary. 1956. "Mathematics and the Existence of Abstract Entities." *Philosophical Studies* 7: 81-88.

Quine, W.V. 1986. "Reply to Charles Parsons." In *The Philosophy of W.V. Quine*, Lewis Edwin Hahn and Paul Arthur Schilpp, eds. La Salle: Open Court, 1986.

Quine, W.V. 1981. "Things and Their Place in Theories." In *Theories and Things*. Cambridge: Harvard University Press, 1981.

Quine, W.V. 1978. "Success and the Limits of Mathematization." In *Theories and Things*. Cambridge: Harvard University Press, 1981.

Quine, W.V. 1974. *The Roots of Reference*. La Salle: The Open Court Press.

Quine, W.V. 1969. "Existence and Quantification." In *Ontological Relativity and Other Essays*. New York: Columbia University Press, 1969.

Quine, W.V. 1960. *Word & Object*. Cambridge: The MIT Press.

Quine, W.V. 1958. "Speaking of Objects." In *Ontological Relativity and Other Essays*. New York: Columbia University Press, 1969.

Quine, W.V. 1955. "Posits and Reality." In *The Ways of Paradox*. Cambridge: Harvard University Press, 1976.

Quine, W.V. 1951. "Two Dogmas of Empiricism." In *From a Logical Point of View*.

Cambridge: Harvard University Press, 1980.

Quine, W.V. 1948. "On What There Is." In *From a Logical Point of View*. Cambridge: Harvard University Press, 1980.

Quine, W.V. 1939. "A Logistical Approach to the Ontological Problem." In *The Ways of Paradox*. Cambridge: Harvard University Press, 1976.

Resnik, Michael. 1997. *Mathematics as a Science of Patterns*. Oxford: Oxford University Press.

Resnik, Michael D. 1993. "A Naturalized Epistemology for a Platonist Mathematical Ontology." In, *Math Worlds: Philosophical and Social Studies of Mathematics and Mathematics Education*. Sal Restivo, et. al., eds. Albany: SUNY Press, 1993.

Shapiro, Stewart. 2000. *Thinking About Mathematics*. Oxford: Oxford University Press.

Shapiro, Stewart. 1993. "Modality and Ontology." Mind 102: 455-481.

Shapiro, Stewart. 1991. *Foundations without Foundationalism: A Case for Second-Order Logic*. Oxford: Oxford University Press.

Sober, Elliott. 1993. "Mathematics and Indispensability." *The Philosophical Review* 102: 35-57.

Van Fraassen, Bas C. 1980. *The Scientific Image*. Oxford: Clarendon Press.