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Kant's Philosophy of Mathematics and the Greek Mathematical Tradition

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Kant made two intimately related claims that greatly influenced the philosophy of mathematics: first, mathematical cognition is synthetic *a priori*; second, mathematical cognition requires intuition for the content and the justification of mathematical concepts and propositions. Kant held that intuitions, like concepts, are a fundamental kind of representation. Intuitions belong (at least for humans) to the faculty of sensibility and represent spatial and temporal properties; concepts belong to the faculty of understanding. Kant contrasts intuitions and concepts by claiming that intuitions are singular representations that relate to objects immediately, while concepts are general representations that relate to objects mediately, that is, mediated by intuitions (A320/B376–77, A68/B93).[1] It is therefore quite natural that some recent accounts of Kant's philosophy of mathematics have focused on the singularity and immediacy of intuition, and have argued that one or both play a central role in Kant's philosophy of mathematics.[2] While not disagreeing with this approach or its fruitfulness, I would like to propose a very different one: I would like to consider the role of intuition in representing magnitudes, and in particular, the spatially extended magnitudes of geometrical constructions. Kant's theory of magnitudes has been largely overlooked; uncovering it complements recent work and gives us a more complete understanding of Kant's philosophy of mathematics. I shall argue that magnitudes are at the heart of Kant's theory of mathematical cognition. In particular, I shall argue that one of the aims of the theory is to explain our cognition of the mathematical properties of magnitudes, for which intuition is indispensable.

Kant's treatment of magnitudes is, I maintain, strongly influenced by the Greek mathematical tradition. That tradition still had currency in Kant's time, allowing Kant to make allusions and tacit references to it. The best evidence for the influence of the Greek mathematical tradition is the strong similarity between it and Kant's own views; I will argue for the influence by bringing out those similarities. More specifically, I will argue that Kant attempted to provide an account of the presuppositions of the Greek conception of magnitudes. I will be emphasizing some of the most basic features of Greek mathematics, features that were entrenched in the time of Euclid and persisted in the mathematical tradition that descended from it. I will therefore turn to Euclid when explaining those features of what I will loosely refer to as the "Greek mathematical tradition" and will not attempt a detailed historical reconstruction of how the Greek mathematical tradition was received in Kant's time.

Before beginning, I would like to make a few general points about the differences between modern and eighteenth-century mathematics that may help orient the reader and explain why Kant's theory of magnitudes has not been fully appreciated. If we think of magnitudes today, we are apt to think of them as abstract quantities that objects *have*. For example, we might think of a walking stick as having a magnitude of four feet in length, a property it can share with other objects. In contrast, Kant thinks of the walking stick as *being* a magnitude. Kant also considers a geometrical figure, such as a triangle, and a particular temporal duration as being, and not merely having, a magnitude. Kant employs several notions of magnitude, but this is the primary notion—something spatially or temporally extended, particular, and relatively concrete.[3] The Greek mathematical tradition shows this same emphasis on concreteness,[4] which marks one of many similarities between it and Kant's account. The difference between Kant's concrete notion of magnitude and the customary understanding of magnitudes today can obscure Kant's meaning.

The Greek mathematical tradition also gave priority to geometry over arithmetic. Painting the Western history of mathematics with a very broad brush, one can say that the dominance of geometry gradually waned from the late Middle Ages through the early modern period but was still influential in the eighteenth century. Numbers gradually moved to center stage in mathematics, and concrete magnitudes came to be treated peripherally. This "arithmetization" of mathematics, continued into the nineteenth century, gradually expanding arithmetic computation and problem solving to include the real numbers, solidifying the emancipation of algebra from geometry, and encouraging an abstract understanding of the calculus. It also led to thinking of space and concrete magnitudes as objects to which numbers can be applied rather than thinking of space as an independent source of mathematical knowledge. The conditions of this application are still considered important today, but applied mathematics is secondary and logically posterior to the development of the mathematics of real numbers.

Moreover, from a modern point of view, foundations begin with arithmetic, and hence we may expect that an account of the role of intuition in mathematical cognition should begin with its role in propositions such as 5 + 7 = 12.

For these reasons, it may look to us as if Kant's references to magnitudes are concerned only with the application of mathematics to objects, and this is how several commentators have treated the Kantian texts on magnitudes.[5] Others have recognized the importance of magnitudes in Kant's account of mathematical cognition more generally;[6] often, however, these accounts focus on Kant's more abstract notions of magnitude in an attempt to untangle his difficult views concerning number and arithmetic.[7] I think, however, that Kant is best understood and explained by focusing first on magnitudes in general and the continuous spatial magnitudes of geometry in particular, and only then considering his views on other continuous magnitudes, discrete magnitudes, arithmetic, and algebra. I will therefore leave a detailed of the latter for another occasion. In this paper, I will argue that the Greek conception of magnitude is at the heart of Kant's philosophy of mathematics and that intuition is required to represent the fundamental mathematical properties of those magnitudes.[8]

The paper has two main parts. Part 1 introduces Kant's definition of magnitude, which I claim invokes the Greek conception of mathematically homogeneous magnitudes. It describes the Greek conception and points of similarity with Kant's own. Kant attempts to account for the mathematical homogeneity of magnitudes by defining what I call strict logical homogeneity. Part I also explains this notion of homogeneity and argues that according to Kant, representing strict logical homogeneity requires intuition.

Part 2 connects the two notions of homogeneity: It argues that in Kant's view, representing mathematically homogeneous magnitudes requires representing a strict logical homogeneity in intuition. More specifically, it argues that Kant wished to account for our cognition of the properties presupposed by the Greek conception of mathematically homogeneous magnitudes: the part-whole composition and equality relations of intuitions.

Part 1: Magnitudes, Homogeneity, and Intuition

1. Magnitude in the Critique of Pure Reason

According to Kant, all objects of human experience conform to conceptual and intuitive conditions. The former are imposed by the categories of the understanding, while the latter are imposed by the pure forms of sensibility—space and time. The combination of these conceptual and intuitive conditions takes the form of principles to which all experience must conform; they are articulated and argued for in the section of the *Critique of Pure Reason* called the System of Principles. The first two sections of the System are the Axioms of Intuition and the Anticipations of Perception. The principle of the Axioms is "All intuitions are extensive magnitudes," and the principle of the Anticipations is "In all appearances the real, which is an object of sensation, has intensive magnitude, i.e., degree" (second edition versions; B202, B207). I will discuss the nature of extensive and intensive magnitudes below. For the moment, what is important is that Kant calls these principles mathematical, by which he does not mean that they are principles of mathematics, that is, that they are themselves included in mathematics. They instead play several roles with respect to magnitudes and mathematics. First, they articulate principles concerning magnitudes that are true of any human experience whatsoever. Second, they explain why and how mathematics applies to the objects of human experience. Third, they make the principles of mathematics possible. The importance of this third role has not always been appreciated, but Kant is relatively clear on this point:[9]

> I will not count among my principles those of mathematics, but will include those upon which the possibility and objective *a priori* validity of the latter are grounded, and which are thus to be regarded as the principle of these principles [*Principium dieser Grundsätze*]. (A160/B199)

Kant reiterates his position in the Discipline of Pure Reason, where he singles out the principle of the Axioms:

> in the Analytic, in the table of principles of pure understanding, I have also thought of certain axioms of intuition; but the principle that was introduced there was not itself an axiom, but only served to provide the principle of the possibility of axioms in general. … For even the possibility of mathematics must be shown in transcendental philosophy. (A733/B761)[10]

We should therefore expect to learn about the relationship between concepts, intuitions, and mathematics in these sections of the *Critique*, and we should expect magnitudes to be at the heart of the matter.

Kant included various additions and clarifications concerning magnitudes in the second edition of the *Critique*. At the beginning of the Axioms, before the argument that all appearances are extensive magnitudes, Kant inserts an argument that all appearances are magnitudes (*simpliciter*). That argument includes a definition of magnitude:

> the consciousness of the manifold homogeneous in intuition in general, insofar as through it the representation of an object first becomes possible, is the concept of magnitude (*Quanti*). (B203)

Kant defines a magnitude as a homogeneous manifold in intuition in general.[11] Kant uses the term 'manifold' in a wide variety of contexts to refer to any sort of multiplicity or muchness whatsoever. Homogeneity, on the other hand, is doing a great deal of work in Kant's definition. Kant provides no explanation of homogeneity following his definition; nonetheless, the notion of a homogeneous magnitude would have been familiar to Kant's readers through its role in the Greek mathematical tradition. An outline of that tradition will help us understand Kant's notions of magnitude and homogeneity and will help explain his account of their relation to mathematics.

2. The Influence of the Greek Mathematical Tradition on Kant's Thought

The Greek conception of homogeneous magnitudes derives from the theory of proportion, whose development is attributed to Eudoxus and is known to us through books 5 and 7 of Euclid's *Elements*.[12] Euclid does not define magnitude [*megathos]*. He does, however, use the term to help pick out the sorts of things that are capable of standing in ratios. Euclid states: "A ratio is a kind of relation with respect to size between two homogeneous [*homogenon*] magnitudes" (bk. 5, def. 3).[13] Euclid has in mind things like lines, plane surfaces, volumes, and numbers; two lines can stand in a ratio, for example, and the numbers 3 and 73 can stand in the ratio 3:73.[14] Thus, lines are homogeneous with lines, planes with planes, and numbers with numbers. A line cannot stand in a ratio to a number, just as an area cannot stand in a ratio to a volume: there is no sense that can be given to the ratio between a particular line segment and the number 5, just as there is no sense that can be given to the ratio between an area and a volume.[15]

Being capable of standing in a ratio is not quite a criterion for homogeneity, however. Euclid leaves open the possibility that two homogeneous magnitudes might not stand in a ratio to one another, for he states that in order to have a ratio each must be capable, when multiplied, of exceeding the other (bk. 5, def. 4). Known as the Archimedean property, it rules out ratios between homogeneous magnitudes if one is infinitesimal or infinitely large with respect to the other. An infinitesimal line segment, for example, does not stand in a ratio to a line segment of 3 inches, even though the two line segments are homogeneous, for no matter how many times the infinitesimal is multiplied, it will never exceed 3 inches. Although it is not necessarily the case that homogeneous magnitudes stand in ratios, what is important about homogeneous magnitudes is that they are the sorts of things that can stand in ratios with each other, and do stand in ratios if we set aside infinitesimal and infinite magnitudes.

We have already seen the way in which homogeneous magnitudes relate to mathematics: they can stand in ratios, and the study of ratios belongs to mathematics. The mathematics of ratios is filled out by the Greek theory of proportions. Homogeneous magnitudes that can stand in ratios can also stand in proportions, which are defined in terms of ratios: "Let magnitudes having the same ratio be called proportional" (bk. 5, def. 6). Euclid defines standing in the same ratio as follows:

> Magnitudes are said to be in the same ratio, first to second and third to fourth, when, equal multiples of the first and third at the same time exceed or at the same time are equal to or at the same time fall short of equal multiplies of the second and the fourth when compared to one another, each to each, whatever multiples are taken. (bk. 5, def. 5)

An anachronistic use of modern algebraic notation helps to bring out the core idea. For any four magnitudes a, b, c, and d and any two positive integers m and n

$$
\begin{aligned}
&a{:}b = c{:}d \text{ iff for all } m, n{:}\\
&ma > nb \rightarrow mc > nd\\
&ma = nb \rightarrow mc = nd\\
&ma < nb \rightarrow mc < nd.
\end{aligned}
$$

In other words, one pair of magnitudes stands in the same ratio as another if the comparative size of the first pair is the same as the comparative size of the second pair under all equimultiple transformations.[16] The definition of sameness of ratios allows that two pairs of magnitudes can stand in the same ratio even if the pairs are inhomogeneous with each other; that is, the magnitudes a and b must be homogeneous with each other, as must c and d, but a and b need not be homogeneous with magnitudes c and d. For example, two lines can stand in the same ratio as the numbers 1 and 5 and in the same ratio as the areas of two triangles. Proportions are useful and powerful in part because they allow us to make claims that relate ratios of different kinds of magnitudes.

Books 5 and 7 of the *Elements* establish sixty-four propositions concerning the ratios and proportions of magnitudes, which include laws governing the ratios between magnitudes under the operations of addition and subtraction, the alteration of terms of ratios, and the addition and subtraction of proportions. These laws are paradigm mathematical laws. They govern the addition and subtraction of numbers, as well as the composition and decomposition of other magnitudes, such as spatial magnitudes. They are therefore more general than the axioms and definitions of geometry, while making an important contribution to it. The mathematical nature of these laws is clear if one considers modern algebraic equivalents of two propositions. Book 5 proposition 1 entails that if magnitudes ma, mb, and mc each consist of m elements of magnitudes a, b, and c respectively, then $ma + mb + mc = m\,(a + b + c)$. Book 5 proposition 18 states that if $a{:}b = c{:}d$, then $(a{+}b){:}b = (c{+}d){:}d$.[17] In short, the Greek theory of ratios and proportions makes explicit many of the mathematical laws to which magnitudes conform, laws that are implicit in the common notions, definitions, and assumptions about magnitudes.[18] I will refer to the totality of those mathematical laws as the "mathematical character" of magnitudes and to the homogeneity of such magnitudes as "mathematical homogeneity." The Greek mathematical tradition incorporated this understanding of the relation between homogeneous magnitudes and mathematics.

Kant's reference to homogeneous magnitudes invokes this aspect of the Greek mathematical tradition, and Kant holds that the theory of ratios and proportions is of central importance. For example, in a letter to Johann Schultz in 1788, Kant states

> Universal arithmetic (algebra) is such an *ampliative* science that one can name none of the rational sciences that are equal to it in this respect; in fact, the remaining parts of pure mathematics expect their growth in large part from the amplification of that universal doctrine of magnitude [Größenlehre]. (10:555)

For Kant, universal arithmetic (algebra) concerned the ratios between magnitudes.[19] Thus, Kant simultaneously stresses the importance of the theory of ratios in mathematics and describes it as a universal doctrine of magnitudes.

Kant also singles out Euclid's theory of magnitudes in an announcement of the program of his lectures for 1765–66. Instead of citing Euclid as an authority on propositions of geometry, as one might expect, Kant cites him as an authority on the doctrine of magnitudes [Größenlehre] (2:307). Kant was not alone in his regard of the Euclidean theory of ratios and its importance for all of mathematics. Wolff, for example, in his entry for *Ratio* in his *Mathematisches Lexicon* (1716, 1170) refers to Euclid's theory of ratios as "indeed the *soul* of the whole of Mathematics [doch die *Seele* der ganzen Mathematick]."[20]

The affinity between the Greek mathematical tradition and Kant's own views should be apparent. The Axioms of Intuition and the Anticipations of Perception contain Kant's explanation of both the applicability of mathematics to objects of experience and the possibility of all mathematical principles. Kant's account rests on magnitudes and their homogeneity and reflects the Greek conception of mathematically homogeneous magnitudes—a conception that formed an influential part of the intellectual heritage in Kant's time. Book 5 of Euclid's *Elements*, which was widely available and read in Kant's day, made this part of the Greek mathematical tradition familiar to Kant's contemporaries.[21] The Greek conception would have been foremost in the minds of his contemporary readers, making it overwhelmingly likely that Kant is invoking it in his own account.[22]

3. Kant on the Representation of Strict Logical Homogeneity

Because the Greek conception of mathematically homogeneous magnitudes shaped Kant's understanding of mathematics, it also set the framework for Kant's views on mathematical cognition. In order to explain the epistemic conditions underlying the Greek conception of mathematical homogeneity, Kant appealed to a different kind of homogeneity, one defined relative to human cognition; it is closely related to a notion of homogeneity that belonged to logic and the study of concepts in Kant's time. Two concepts or things are homogeneous with respect to a concept if they both fall under that concept. For example, the concept of a Clydesdale and the concept of a Shetland are homogeneous with respect to the concept horse.[23] I will call this "logical homogeneity." In the logic of Wolff and his followers as well as in Kant, concepts can be ordered into a genus-species hierarchy. Concepts or things can be more or less logically homogeneous with each other depending upon how general the common concept under which they fall is, and hence logical homogeneity comes in degrees. For example, a Clydesdale is more homogeneous with a Shetland than it is with a jaguar.

Kant uses the notion of logical homogeneity in his defense of a principle of reason concerning the systematic unity of experience, a principle of the homogeneity of forms (A651–63/B679–91). It was also commonly thought to be a requirement of counting that the objects counted fall under a common concept.[24] More importantly, however, Kant uses a version of logical homogeneity to relate mathematical homogeneity to our cognitive abilities.

In his lectures on metaphysics, Kant uses logical homogeneity to contrast a *quantum* with a *compositum*. In the Axioms, he distinguishes between two sorts of magnitude, *quanta* and *quantitas*; the former is a concrete magnitude, the latter a more abstract counterpart. While the distinction is crucial elsewhere, it is not crucial for our present point, and *quanta* can be taken for now to be synonymous with magnitude.[25] Kant states that both a *quantum* and a *compositum* contain a plurality, but a *compositum* allows for an aggregate of heterogeneous parts, while a *quantum* requires homogeneity among the parts. Kant articulates the homogeneity requirement of *quanta* as follows:

> *Homogeneitatem*, i.e. things from one and the same genus (*genus*) [Gattung (*genus*)], hence *compositum* differs from *quantum*, and the many would in that case be able to be a variety [varietaet], every *quantum* contains a multitude [Menge] but not every multitude is a *quantum*; rather, it is only when the parts are homogeneous. (29:990, 1794–95)

Thus, a *quantum* requires that the manifold at least be logically homogeneous.

Kant next explains his use of genus and species in this context. He introduces the terms quiddity, quality, and quantity, corresponding to a once familiar trio of questions one can ask about a thing: What? What sort? How much? The contrast between quiddity and quality turns on specific differences:

> *Quidditas*, if one wants to put it that way, would be distinguished from quality as the determination of genus and the specific difference; e.g., *quiddity* the genus of which is *essence*: but whether it is hard or soft belongs

> to quality, therefore in regard to species conceived under the genus. (29:991, 1794–95)

Quiddity consists of the genus and the specific difference that together define the essence, while quality concerns further specific differences. Kant goes on to contrast quality and quantity:

> quality differs from quantity in that, and to the extent that, the [former][26] indicates something in the same object which is inhomogeneous [ungleichartiges] with regard to other determinations found in it. Therefore quality is that determination of a thing according to which whatever is specifically different finds itself under the same genus, and can be distinguished from it. This is heterogeneous [heterogen] in contrast to that which is not specifically different, or to the homogeneous [homogen]. (29:992, 1794–95)[27]

Qualities are specifically different characteristics, and these specific differences are heterogeneous to each other. Quantity, in contrast, does not even allow specific differences, and the lack of specific difference is called homogeneity (see also 29:839, 1782–83). In short, qualitative differences are heterogeneous, and the homogeneous excludes any qualitative difference at all. I will call the logical homogeneity that excludes all qualitative difference "maximum" or "strict" logical homogeneity.

In further notes from his lectures, Kant explains the notion of strict logical homogeneity in terms of numerical difference. We have seen that the distinction between quiddity, quality, and quantity corresponds to generic and essential specific difference, mere specific difference, and the not-even-specifically different. Kant claims in other lectures on metaphysics that all difference is generic, specific, or numerical (28:422, 1784–85; 28:561, 1790–1). He adds in one lecture that "two drops of water on 2 needle points are numerically different and specifically identical" (28:422, 1784–85). Kant elaborates on the notion of numerical difference in a lecture on metaphysics most likely delivered in the decade of the *Critique*:

> The concept of a man [Mann] is already more closely determined than the concept of a human [Menschen]; that is the case in relation to every genus and every species, in which this species can again become a genus with respect to another species. The genus differentiates itself from a species, in so far as different species can be contained under a genus. These under a genus are called inferior concepts [*conceptus inferiores*] and their difference is specific difference [*differentia specifica*]. If this species cannot itself again be regarded as a genus, then it is a lowest species [*species infi-*

> *mae*] and the difference of multiple [mehrerer] lowest species is numerical difference [*Differentia numerica*]. (28:504, late 1780s)

There are two points here. First, a concept's status as either a genus or species depends upon whether it is viewed in relation to a concept that falls under it or in relation to a concept under which it falls. Thus, concepts represent both generic differences of quiddity and specific differences of quality, and both quiddity and quality can be described more broadly as specific differences. Second, an *infima species* is a concept under which no further concepts can fall, and hence is a species but not a genus. Numerical difference is difference even where no further specific difference is possible.[28]

Kant's position implies that a *quantum* differs from a *compositum* in being homogeneous, that being homogeneous corresponds to quantity, and that homogeneity requires specific identity with numerical diversity. Kant is explicit about this result.

> Homogeneity is specific identity with numerical diversity [numerischen Diversitaet], and a *quantum* consists of homogeneous parts [partibus homogeneis]. (28:504, late 1780s)

Quanta, that is, concrete magnitudes, exhibit maximal logical homogeneity, that is, the parts are specifically identical yet numerically diverse.

What is crucial here is that in Kant's view, concepts on their own can only represent qualities, that is, specific differences; they cannot represent bare numerical difference. In contrast, intuition can represent bare numerical difference. Hence, intuition makes it possible to represent a strictly homogeneous manifold. Furthermore, Kant explicitly ties the definition of strict logical homogeneity to the mathematical homogeneity of concrete magnitudes (*quanta*). Hence, concepts alone cannot represent concrete magnitudes, while intuition makes it possible to represent them.[29]

Since the only forms of intuition for us are space and time, the argument so far establishes that space or time make it possible for us to represent magnitudes. Space and time, however, are not the only kinds of magnitudes or the only strictly homogeneous manifolds there are. Kant also holds that an intensive magnitude, such as the intensity of a light, is a magnitude and contains a strict logical homogeneity. Nevertheless, Kant holds that we cannot even represent an intensive magnitude as a magnitude at all (as containing a homogeneous manifold at all) without the aid of space or time. (I will discuss intensive magnitudes in more detail in section 6 below.)[30] Consequently, intuition is not just a sufficient condition for representing magnitudes; it is a necessary condition. Kant holds that concepts alone cannot represent concrete magnitudes and that intuition is required to represent them.

So far, I have explained Kant's conception of magnitude and argued that Kant's understanding of mathematics is strongly influenced by the Greek conception of magnitudes. Under the influence of the Greek mathematical tradition, Kant thinks that magnitudes are at the basis of mathematics and the mathematical character of experience, and as a consequence he thinks that our representation of magnitudes is at the heart of mathematical cognition. I have also shown why Kant thinks intuition is required to represent magnitudes and hence is required for mathematical cognition: intuition is required to represent a strict logical homogeneity, that is, a bare numerical difference. Nevertheless, Kant's argument does not explain what strict logical homogeneity has to do with mathematical homogeneity in the Greek sense, and hence with mathematics. I will take up this issue in part 2. First, however, I will give further evidence for our results so far.

4. Numerical Diversity, the Limitations of Conceptual Representation, and Intuition

The argument I have outlined turns on the limitations of conceptual representation and the role of intuition in overcoming them. Since I have uncovered the argument using works that were not published by Kant, confirmation from Kant's published writings would be welcome. The Amphiboly of Pure Reason in the *Critique* provides it.

In the Amphiboly, Kant attacks Leibniz's principle of the identity of indiscernibles, which claims that two completely indiscernible individuals are identical. According to Leibniz,

> [t]here is no such thing as two individuals indiscernible from each other. … Two drops of water, or milk, viewed with a microscope, will appear distinguishable from each other. (1969, 700)

Despite Leibniz's claim that two drops of water or milk can be distinguished through a microscope, his point is not that there will always be differences among individuals that we can empirically verify. Leibniz believes in an underlying rational order of the universe that reflects God's intellect. He believes that there must be a sufficient reason for every fact, and that all reasons correspond to subject-predicate relations between concepts. Leibniz's point in this passage is that God, if no one else, can always distinguish any two individuals by their qualities, no matter how similar they may appear to be, so that in principle, distinct individuals are always conceptually distinguishable. Leibniz's position, as Benson Mates puts it, is that

> God's concepts … are fine-grained enough to distinguish each individual from all the others. It is obvious that by virtue of their accidents, any two individuals will fall together under a very large number of concepts, that is, will have a large number of attributes in common. But the principle assures us that however similar they may be, there will always be some concept under which one of them falls and the other does not. (Mates 1986, 135)

Leibniz's position can be recast in Kantian terms: between distinct individuals there are always specific differences that can be represented conceptually. It is this claim that Kant attacks in the Amphiboly. Kant claims that the role of intuition in our cognition allows for numerical difference even when two objects are conceptually indistinguishable:

> However identical everything may be in regard to [the comparison of two objects in respect of their concepts], the difference of the places of these appearances at the same time is still an adequate ground for the **numerical difference** of the object (of the senses) itself. Thus, in the case of two drops of water one can completely abstract from all inner difference (of quality and quantity), and it is enough that they be intuited in different places at the same time in order for them to be held numerically different. (A263–64/B319–20)

Kant reverses Leibniz's claim concerning two drops of water, and argues that if we abstract from all inner qualitative and quantitative differences, their simultaneous location in different regions of space is sufficient to ground their (discrete) numerical diversity.

Kant diagnoses the source of Leibniz's error as a misunderstanding of the nature of human cognition. Leibniz, he claims, fails to recognize that we have a faculty of sensibility with its own distinctive kind of representation, namely, intuition. Leibniz believes we only have conceptual representations, the sort that belong to the intellectual faculty, and that sensation is but a confused form of conceptual representation. According to Kant, Leibniz in effect assimilates intuitive representation into his model of conceptual representation. Since Leibniz thinks that objects of experience are cognized only through the understanding, all representation is of a fundamentally conceptual nature. Hence, the difference between any two individuals can only be represented conceptually as a specific difference, a difference in quality.

> Leibniz took the appearances [i.e., objects of sensibility] … for *intelligibilia*, i.e., objects of the pure understanding … and on that assumption his principle of the identity of indiscernibles … certainly could not be disputed. But since they are objects of sensibility … plurality and numerical difference are already given us by space itself. … For one part of space, although completely similar and equal to another part, is still outside the other and for this very reason is a different part from that which abuts it to constitute a greater space. (A264/B320)

Space, the form of the faculty of sensibility, is a source of plurality and numerical difference, and can give numerical diversity to specifically identical individuals.

Our interest is not in whether his argument against Leibniz is fair or his diagnosis correct, but in Kant's claim that if objects of experience were objects of understanding alone, then the principle of the identity of indiscernibles would hold (see also A272/B328). In other words, if the only means for us to represent objects were conceptual, then we could not represent specific identity with numerical diversity. Furthermore, it is intuition that allows us to overcome this limitation of conceptual representation.[31] Thus, intuition is required to represent strict logical homogeneity.

The Amphiboly reinforces the claim that intuition is required to represent bare numerical difference and supports the argument given at the end of section 3. Nevertheless, we have not yet seen what strict logical homogeneity has to do with mathematical homogeneity and hence mathematics. The remainder of this paper focuses on that explanation.

Part 2: Composition, Parts and Wholes, and Equality

Part 2 will argue that Kant aimed to explain the presuppositions underlying the Greek conception of magnitudes. It is more reconstructive, and the evidence for it is less direct than that for part 1. Kant does not explicitly articulate his aims, at least in part because the Greek mathematical tradition was familiar to his contemporaries. The strongest evidence is based on an examination of the cognitive presuppositions of the Greek conception (section 5) and an analysis of Kant's account of those presuppositions (sections 6–8). The following analysis reflects the way those presuppositions might appear to someone strongly influenced by the Greek mathematical tradition rather than the way they might appear to us today.

5. Presuppositions of the Greek Conception of Magnitudes

As explained above, the Greek conception of the mathematical character of magnitudes rests on the fact that homogeneous magnitudes stand in ratios, and magnitudes are said to have a ratio to one another that are capable, when multiplied, of exceeding one another (book 5, def. 4). The Greek notion of ratio is in turn based on the presumption that magnitudes can be composed of multiples of other magnitudes, and stand in relations of smaller, equal, and larger. The definition of sameness of ratio—that is, being in proportion—makes the same presumptions: having the same ratio requires that the relations of smaller, equal, and larger be invariant under all equimultiple compositions. These presumptions are found throughout books 5 and 7, and are exemplified by book 5, proposition 1, cited above: if magnitudes ma, mb, and mc each consist of m elements of magnitudes a, b and c, respectively, then $ma + mb + mc = m\ (a + b + c)$.[32]

The multiplication of magnitudes assumes that the composed magnitudes are not only disjoint but equal; however, composition out of equal parts is simply a special case of composition. Composition of a whole magnitude out of possibly unequal parts is exemplified by book 5, proposition 18, cited above: if $a{:}b = c{:}d$, then $(a+b){:}b = (c+d){:}d$. In general, the composition of magnitudes and the part-whole relations between the parts and the wholes they compose are fundamental to Greek mathematics.

Because Euclid introduces the notion of homogeneous magnitudes in the definition of ratio, and two magnitudes having a ratio is defined by appeal to composition, what counts as mathematically homogeneous in the Greek sense depends directly upon the understanding of what can be composed. In fact, the notions of homogeneity and composition are interdependent.[33] A closer look at the Greek notion of composition will fill out their notion of mathematical homogeneity.

Composition is not just any sort of putting together of magnitudes; it is a putting together of homogeneous magnitudes that yields more of the same kind, that is, it yields something homogeneous with the composed magnitudes while being more than the composed magnitudes. There are two important restrictions on this Greek notion of composition. First, there is no sense in which one can compose inhomogeneous magnitudes. A line and a plane, for example, cannot be composed together. One can draw a line intersecting a plane, but that does not count as a composition of the two.[34] Second, homogeneous magnitudes cannot be composed in such a way as to yield a magnitude inhomogeneous with them. For example, a line segment can be composed with another line segment only in such a way as to yield a longer line segment; there is no way to compose them in such a way that yields a point, plane, or solid. Reflection on some disallowed cases is helpful in understanding this restriction. One might, for example, "construct" a point as the intersection of two lines and count that as "composition" of the point. This would not count as composition in the Greek sense, however. Another case is perhaps more in keeping with the meaning of composition: one might think that lines could compose a square or cube by outlining these figures. This also would not count as composition in the Greek sense; while the lines can be so composed, the square or cube are only delimited, not "composed" out of them—that is, the parts of the lines do not constitute or make up the enclosed volume. Finally, one might think that parallel lines could compose a plane. In the Greek view, however, lines have no width, and no amount of composition of this sort will yield any width and hence a plane.[35]

These are descriptions of what composition does not include. The restrictions are a reflection of a positive view of composition, intended to capture the idea that certain things of the same sort can, when put together, yield *more* of the same kind of thing. Two regions of space, for example, can compose a larger space. Common notion 5 of the *Elements* states that the whole is greater than its parts, and this was held to be true of a whole composed of homogeneous parts.[36] The important point is that composition yields magnitudes that are larger and nevertheless homogeneous with what one started.[37]

This notion of composition underlying the Greek theory of proportions also underlies measurement. On the most basic understanding, measuring requires that we be able to compose multiples equal to some magnitude taken as a unit and that we be able to make comparative judgments between the measure and the measured.[38] Thus, in the Greek mathematical tradition, the mathematical properties of magnitudes and our ability to apply numbers to magnitudes in measurement rest upon the same minimal assumptions.[39]

So far, I have focused on the presuppositions of the Greek conception of magnitudes that are more or less explicit in Euclid: composition, part/whole relations, and comparative relations of less than, equal to, and greater than. We can reduce the number of presuppositions, since less and greater can be defined by stipulating that one magnitude is less than another if the first is equal to a proper part of the second, and greater than it if the second is equal to a proper part of the first.[40] Thus, on a rather straightforward analysis, the Greek theory of ratios and proportions is fundamentally mereological: the minimal basis assumed by the Greek theory of ratios and proportions is that mathematically homogeneous magnitudes can stand in part-whole composition relations and the equality relation.

These minimal assumptions can also be viewed from an epistemological standpoint. Our cognition of the mathematical character of magnitudes depends, at a minimum, on our ability to make judgments about the part-whole composition of homogeneous magnitudes and our ability to make judgments of equality between homogeneous magnitudes. The remainder of part 2 will argue that Kant's theory of mathematical cognition attempts to account for these presuppositions. I will do so by arguing that in Kant's view intuition is required to cognize the part-whole composition of magnitudes.

6. The Part-Whole Composition of Magnitudes

Kant defines and distinguishes extensive and intensive magnitudes using the part-whole relation. Kant defines extensive magnitude as that magnitude "in which the representation of the parts makes possible the representation of the whole (and thus necessarily precedes it)" (A162/B203). The paradigm example of an extensive magnitude is a determinate region of space—that is, a region of space such as a line, a circle or a cube. In contrast, an intensive magnitude contains a manifold, but the apprehension of it "does not proceed from the parts to the whole" (A168/B210).[41] A paradigm example of an intensive magnitude is the intensity of a light. Kant holds that because the apprehension does not proceed from the parts to the whole, we apprehend it as a unity. Kant's point is that in apprehending an intensive magnitude, we do not directly represent its part-whole structure. We do not, for example, apprehend various degrees of intensity of a light, only its total intensity. In contrast, whenever we apprehend an extensive magnitude, we also apprehend its part-whole relations. I cannot, for example, apprehend a determinate region of space without thereby apprehending its parts.[42]

Extensive magnitudes play the leading role in Kant's philosophy of mathematics. Kant states of the principle of the Axioms, which concerns extensive and not intensive magnitudes, that it

> greatly extends our cognition. For it is that alone, which makes pure mathematics in its entire precision applicable to the objects of experience. (A165/B206)

Kant holds that because the representation of an intensive magnitude is of a unity, we can apprehend its part-whole structure only indirectly, with the aid of extensive magnitude. In fact, we depend upon extensive magnitudes to represent them as containing a strictly homogeneous manifold, and hence to represent them as magnitudes at all. We become aware that an intensive magnitude is a strictly homogeneous manifold by representing the intensity of a light, for example, diminishing down to zero, or increasing from zero up to a given intensity (B208, A168/B210). Thus, the manifoldness and the part-whole structure of the intensity of a light reveals itself only by representing the parts of that manifold by means of the extensive magnitude of time. Extensive magnitudes play a leading role in Kant's philosophy of mathematics precisely because they manifest their part-whole structure in a way that makes that part-whole structure cognitively accessible. The most important property of magnitudes, after homogeneity, is their part-whole structure.[43]

Kant does not directly discuss the concepts of part and whole in the *Critique* or other published work. Nevertheless, Kant's lectures on metaphysics and his notes reveal two important features of Kant's views on the part-whole relation. First, the part-whole relation corresponds to the categories of quantity outlined in the *Critique*. Second, the homogeneity of intuition is required not only to represent magnitudes, but also to represent the part-whole composition of magnitudes.

The connection between part-whole relations and the categories of quantity—unity, plurality, and totality—are reflected in the development of Kant's critical philosophy. Alexander Baumgarten, a student of Wolff, wrote a metaphysics text that Kant used for many years in his lectures. Kant repeatedly worked through this Leibnizian-Wolffian metaphysics, considering and reconsidering the fundamental concepts upon which it was built; his evolving views are revealed in student lecture notes, Kant's notes in his Baumgarten text, and various other notes (see 17:1–745). Kant discusses various pairs and triplets of concepts and their relations, but none so frequently as one-many-one. His discussions show that he thought of these concepts as corresponding to the concepts of unity-plurality-totality and to the concepts of part and whole. The three categories correspond to the part-whole relation by giving us the concept of the unity of a part, the plurality of parts, and the totality of parts in a whole. In a metaphysics lecture delivered in 1784–85, for example, Kant explicitly states that the concept of part and whole stands under the categories of quantity (28:423; see also 29:803, 1782–83; 28:504–5, late 1780s). Thus, the three categories of quantity allow us to cognize part-whole relations.[44]

More specifically, these categories allow us to cognize the part-whole relations of magnitudes. Kant describes the part-whole relations of magnitudes using the categories, as in this passage from the *Critique of Judgment*:

> That something is a magnitude (*quantum*) can be cognized from the thing itself without all comparison with another: if, namely, a plurality of the homogeneous together makes a unity. (5:248).

This role for the categories of quantity is further confirmed by the way in which Kant refers to them. In Kant's account, after homogeneity the most important property of magnitudes is their part-whole composition. Kant often refers to the categories of quantity as the categories of magnitude or the category of magnitude (B115, B162, B193, B201).[45]

The role of the categories of quantity supports the view that Kant's approach to mathematics is markedly different from our modern set-theoretic approach. Kant's is fundamentally mereological—another way in which Kant's account follows the Greek mathematical tradition found in Euclid.

So far, we have established that the categories of quantity allow us to cognize the part-whole relations of magnitudes. As we saw in part 1, it follows from Kant's definition of magnitude and his definition of homogeneity that the representation of magnitudes requires intuition. Kant also holds that intuition is required for the representation of the part-whole composition of magnitudes. A good number of the notes in which Kant discusses the one-many-one and unity-plurality-totality concepts also concern the role of intuition. For example, lecture notes from a course on metaphysics state the following:

> The concept of magnitude is properly characteristic of understanding [zu dem Verstand gerade zu eigen] because it concerns itself with the connection of the manifold homogeneous [mannigfaltige gleichartigen]. Magnitude [Größe] is employed in mathematics through the help of a pure intuition in sensibility, i.e., through the form of space and time in the determination of each figure or number. But in philosophy it cannot be determined from the concept alone whether the category of magnitude [Größe] has objective reality. I.e., it cannot be cognized that many

> together constitute a one [daß Vieles zusammen Eins ausmache]. (29:992, 1794–95)

This passage, which concerns the role of magnitudes in mathematics, states that intuition allows us to establish the objective reality of the category of magnitude, that is, allows us to cognize that a many constitutes a one. In other words, intuition allows us to cognize that parts compose a whole. Kant's point is that there is a special sort of composition particular to the part-whole relations of magnitudes, and that intuition is required in order to cognize this composition.[46]

For humans, the two forms of intuition are space and time, so it is not surprising that Kant considered the role of each in representing the relations between parts and wholes. For example, Kant entertained the view that the temporal act of drawing a line successively represents the parts of a line, while the spatial figure of the line simultaneously represents all those parts, and hence represents them as coexisting in a whole.[47] In several important passages of the *Critique*, Kant refers to drawing a line in thought through a figurative synthesis. These passages are evidence of Kant's kinematic conception of mathematics, to which he appeals in response to problems in the foundations of the calculus.[48] Kant's suggestion about the particular roles of space and time would neatly unify his views on the part-whole relations of mathematically homogeneous magnitudes and his kinematic conception of mathematics. Regardless whether Kant was committed to these particular roles for space in time in representing part-whole relations, which I think quite possible, he often mentions the role of space and time in simply representing homogeneous parts:

> the category of magnitude [Größe], as a homogeneous many that together constitutes [ausmacht] one; this cannot be grasped without space and time. (29:979)

Space and time allow us to grasp the category of magnitude by allowing us to grasp that a homogeneous many together constitute a one, that is, that the homogeneous parts constitute a whole. A note Kant wrote in the margin of the Axioms of Intuition (in his copy of the first edition of the *Critique*) emphasizes that the important property of space and time is their homogeneity:

> We can never take up a manifold as such in perception without doing so in space and time. But since we do not intuit these for themselves, we must take up the homogeneous manifold in general in accordance with the concepts of magnitude. (23:29)

The *Critique* provides further important evidence that intuition plays a role in representing the composition of magnitudes. As mentioned in section 1, Kant calls the Axioms of Intuition and the Anticipations of Perception mathematical principles. In the second edition, Kant added a footnote to further explain what makes these principles relevant to mathematics. He delineates various sorts of synthesis or combination [Verbindung]; the combination at the root of the mathematical principles is composition [Zusammensetzung] (B201 n. 1, 4:343).[49] This special act of synthesis is employed *only* in the representation of magnitudes, and its distinguishing feature is that it synthesizes a homogeneous manifold:

> All **combination** (conjunctio) is either **composition** (compositio) or **connection** (nexus). The former is the synthesis of a manifold of what **does not necessarily** belong **to each other**, as e.g. the two triangles into which a square is divided by the diagonal do not of themselves necessarily belong to each other, and of such a sort is the synthesis of the **homogeneous** in everything that can be considered **mathematically.** (B201 n. 1)

The synthesis of composition of a strictly logically homogeneous manifold generates our representations of magnitudes.[50] This and previous passages show that by representing a strictly logically homogeneous manifold, intuition makes possible a special synthesis of composition underlying our cognition of "everything that can be considered mathematically."

We saw in the previous section that in the Greek tradition, the notion of composition and the notion of mathematically homogenous magnitudes are interdependent. We have established that in Kant's view, there is a fundamental connection between his notion of strictly logically homogeneous magnitudes and composition, and that this composition underlies mathematical cognition. The parallel is too striking to be accidental, and provides mounting evidence that Kant's approach to mathematical cognition is fundamentally shaped by the Greek mathematical tradition. But what is the connection between strict logical homogeneity and the Greek mathematical homogeneity? Stated differently, what does Kant think is special about the composition of a strict logical homogeneity, and what does it have to do with mathematics?

7. Intuition and the Composition of Magnitudes

Kant's understanding of intuition's role in the representation of composition is best brought out by contrasting it to Kant's understanding of conceptual representation. In Kant's view, the intension of a concept consists of any constituent concepts it has; the concept 'human,' for example, might contain the concepts 'animal' and 'rational.' Kant calls constituent concepts 'partial' concepts [*Teilbegriffe*], and the relation between a concept and its partial concepts is that of whole to part. Kant's notion of intension is familiar, but his notion of extension is not. In the modern view, an extension consists of the objects that fall under a concept. Kant holds that objects fall under concepts, but he also holds that concepts fall under concepts, and in his lectures on logic, Kant describes the logical extension of a concept as the concepts that fall under a concept (9:98). The concept 'human', for example, is part of the extension of the concept 'animal'. The part-whole relation between concepts and their intensions is reciprocal to that between concepts and their extensions; the concept 'animal' is part of the intention of the concept 'human' if and only if the latter is part of the extension of the former.

As explained in section 3 above, Kant holds that concepts represent qualitative differences, that is, specific differences. If we combine the concept 'rational' with the concept 'animal', we generate a more specific concept and restrict the extension of the concept 'animal' to a smaller extension. We also saw that in Kant's view, concepts can represent *only* qualitative differences; hence, the combination of unique concepts always results in ever more specific concepts and ever smaller extensions. There is no room in this theory for acombination of concepts that would yield the representation of *more* than what the constituent concepts represent.

In contrast, the composition of a strictly logically homogeneous manifold, of bare numerical difference without specific difference, yields *more* of the same. Kant makes the importance of this property of intuition clear in a passage from the Amphiboly already cited:

> Leibniz took the appearances [i.e. objects of sensibility] … for *intelligibilia*, i.e., objects of the pure understanding … and on that assumption his principle of the identity of indiscernibles … certainly could not be disputed. But since they are objects of sensibility … plurality and numerical difference are already given us by space itself. … For one part of space, although completely similar and equal to another part, is still outside the other and for this very reason is a different part from that which abuts it [*zu ihm hinzukommt*] to constitute a greater space. (A264/B320)

Kant states that parts of space are numerically different while being similar and equal, and then adds that adjoining parts of space together constitute a greater space. This latter property is not directly relevant to the point of his argument, which is that space allows the representation of bare numerical difference. Kant mentions it because he thinks that the compositionality property of space is intimately tied to bare numerical difference.

Kant holds that the composition of parts that have strict logical homogeneity results in a whole that is greater than those parts. Concepts on their own, no matter how they are combined, do not allow the representation of this kind of part-whole composition. And it is exactly this sort of composition that Kant thinks is characteristic of "that which can be considered mathematically." Without the representation of this sort of combination, mathematical cognition would not be possible at all.[51]

The inability of concepts to represent mathematical composition can be brought out in another way. As discussed in section 4 above, Kant claims that if objects were objects of understanding alone and hence represented only by means of concepts, then Leibniz's principle of the identity of indiscernibles would hold. Kant uses this hypothetical to bring out the limitation of conceptual representation. If we attempted to use only concepts to represent the composition of strictly homogeneous units of space into a larger space, we would have to represent each of the spaces as instances of one and the same concept, such as the concept 'cubic foot'. In that case, however, each instance of the concept would be indistinguishable and hence identical, which means we could at best succeed in repeatedly picking out one and the same object. Kant makes this point in *What Progress has Metaphysics Made Since the Time of Leibniz and Wolff?*

> According to mere concepts of the understanding, it is a contradiction to think of two things outside of each other that are nevertheless fully identical in respect of all their inner determinations (of quality and quantity); it is always one and the same thing thought twice (numerically one). (20:280, cf. A263/B319 and A282/B338)

In other words, concepts alone cannot represent pure numerical diversity, and our attempt at representing distinct spaces collapses into representing just one. As Kant puts it, conceptual representation alone would "bring the whole of infinite space into a cubic inch and less" (20:282). Since concepts alone cannot represent the bare numerical diversity of mathematical composition, we must rely on intuition.

Kant's views on the nature of intuitive and conceptual representation forge a link between strict logical homogeneity and Greek mathematical homogeneity. Both are closely tied to the special sort of composition that, according to the Greek mathematical tradition, underlies mathematics. A strict logical homogeneity is required to represent that composition, and hence intuition is required for mathematical cognition.[52]

8. Part-Whole Composition, Equality, and the Mathematics of Magnitudes

Part 1 established that Kant thought that intuition is required to represent magnitudes, and now we have seen why Kant thinks it is required to represent their mathematical properties. There is nevertheless more to the mathematical character of magnitudes than simply being composable into more of the same kind. If Kant aims to account for our mathematical cognition by accounting for our cognition of magnitudes, one would like to know how he would account for these further mathematical properties.

Kant holds that a whole is greater than its parts.[53] A composed magnitude will therefore be larger than any of its parts, which imposes an ordering on magnitudes. This ordering is severely limited in two ways, however. First, it will order magnitudes only in relation to fully contained parts; it establishes that a magnitude is smaller than a second magnitude of which it is a fully contained part, that the second magnitude is smaller than a third of which it is a fully contained part, and so on. It does not, however, order the relative sizes of magnitudes that are disjoint or partially overlapping; the ordering based on part-whole composition is a partial rather than a total ordering restricted to full part-whole containment. Second, even within the partial ordering of parts and subparts, all that is established is that one magnitude is larger than another; the ordering does not determine how much greater the whole is than any of its parts. It does not, for example, determine whether one magnitude is twice as large as one of its parts or whether the two magnitudes stand in some other ratio.[54] Thus, the part-whole composition relation of a strict logical homogeneity falls far short of establishing the kinds of relations between magnitudes that are established in books 5 and 7 of Euclid's *Elements*.

As we saw in section 5 above, the theory of proportions in the Greek mathematical tradition presupposes that magnitudes stand in comparative size relations. We also saw that the minimal basis presupposed by the mathematical properties of magnitudes described in Euclid's books 5 and 7 consists of the part-whole composition relation and the relation of equality. That is, if the part-whole composition of magnitudes is supplemented with the relation of equality, then the remaining mathematical properties of magnitudes follow, because greater and less can be defined using equality and the part-whole relation. Using equality in this way expands the greater- and less-than relations into a total ordering on magnitudes.

Kant introduces the relation of equality into his theory of magnitudes in just this way. Kant uses equality to define a size relation between distinct magnitudes that are not related by part-whole composition. Kant thereby gives a sense to one magnitude being greater or less than another even when the magnitudes are only partially overlapping or do not overlap at all. In an extended lecture note on the part-whole relations of magnitudes, Kant defines larger and smaller for magnitudes using the relation of equality as follows:

> A > than B if a part of A = B; in contrast A < B, if A is equal to a part of B. (28:506, late 1780s)[55]

Thus, one magnitude will be larger than another as long as a part of it is equal in size to the other, which in principle allows the ordering of *any* two magnitudes of the same kind according to their size.[56]

The relation of equality also allows magnitudes to stand in ratios, pairs of magnitudes to stand in the same ratio, and magnitudes to measure one another. As I noted in section 2 above, the *Elements* state that two magnitudes have a ratio to one another that are capable, when multiplied, of exceeding one another, and that two magnitude pairs stand in the same ratio when the relations of equality, greater than, and less than remain the same under equimultiple transformations. The multiplication of magnitudes rests on the composition of multiple distinct magnitudes equal to a given magnitude, which can be composed to form a whole. These are the same requirements that underlie measurement, which rests on the stipulation of a unit magnitude, the composing of magnitudes equal to the unit, and the equality of the composed and measured magnitudes. Thus, Kant's use of the relation of equality accounts for the remaining properties presupposed by the Greek theory of magnitudes.[57] Kant's theory of mathematical cognition rests on the representation of a strict logical homogeneity in intuition, the categories of quantity, and the concept of equality.

There is a final point I would like to address before closing. On the interpretation I have given, the representation of part-whole representations of magnitudes in intuition is required to make mathematical cognition possible. There is, however, another use of part-whole relations in Kant that might suggest a merely logical foundation for mathematical relations that would not require intuition: the part-whole relations between the extensions of concepts.

In Kant's view, both things and concepts fall under a concept, but from a purely logical point of view, which considers the relation between concepts in abstraction from their relation to objects, the extension of a concept consists of the (possible) concepts that fall under it. Kant explains various logical forms of judgment in terms of the relations between the extensions of concepts and uses Euler's circles (Venn diagrams) to illustrate them (9:109; he sometimes uses squares). For example, the "All perrisodactyla are ungulates" would be illustrated by a circle representing the concept 'perrisodactyl' fully contained within a larger circle represented by 'ungulate'. Strikingly, Kant also draws analogies between the part-whole relations of concept extensions and the the part-whole relations of space.[58] Kant's views suggest that the part-whole relations between concepts could somehow account for mathematical relations in the manner suggested for the part-whole relations of magnitudes.

This purely logical approach will not work, however. As we saw above, the extensions of concepts consist of other concepts that represent specific differences and hence heterogeneous manifolds, and in Kant's view such manifolds cannot be joined together in a manner that satisfies the Greek conception of composition.

Even if this problem were set aside, however, in Kant's view, concept extensions cannot stand in the relation of equality. (Kant does state that distinct concepts can have identical extensions (9:98), but he nowhere allows that extensions could be distinct yet equal.) As a consequence, equality cannot be used to supplement the comparative size relations between extensions to form what we would call a total ordering.

In fact, Kant explicitly mentions this limitation of the part-whole relations between concept extensions:

> One concept is not *broader* than another because it contains *more* under itself—for one cannot know that—but rather insofar as it contains under itself the *other concept* and *besides this still more.* (9:98)

This passage shows that despite his use of Venn diagrams to illustrate logical relations, Kant is well aware, and even goes out of his way to state, that the comparative size between concept extensions is only what we would call a partial relation, restricted to full part-whole containment. Since Kant does not allow the relation of equality between extensions, there is no sense that can be given to the ratios between extensions, not even when one is fully contained in another, and there is also no way to introduce a measure of size.

9. Conclusion

I have argued that the strong similarities between the Greek conception of magnitudes and Kant's treatment of them show that Kant developed his theory of mathematical cognition in response to the Greek mathematical tradition. This influence on his views is not immediately apparent, since Kant could rely on his reader's familiarity with the Greek conception of magnitudes and the Eudoxian theory of proportions. Furthermore, Kant does not present his full theory of mathematical cognition in the *Critique of Pure Reason*, which had broader aims. Nevertheless, the mathematical principles of the System of Principles place magnitudes at the center of his theory. Both published and unpublished texts show that Kant closely followed the Eudoxian theory in thinking that the mathematics of magnitudes rests on the composition of magnitudes that are mathematically homogeneous. Kant therefore wished to explain our cognition of this composition and of mathematical homogeneity. Kant also followed the Greek mathematical tradition in his mereological approach to our cognition of magnitudes. The resulting theory of cognition posits, on the one hand, a role for intuition in cognizing mathematical composition, and on the other, a role for the categories of quantity in cognizing the part-whole relations among magnitudes. If we also take into account Kant's use of the concept of equality, we can see that he provides a foundation for the Eudoxian theory of proportions as well as the theory of measurement.

If Kant had not followed the Euclidean tradition, had he thought of arithmetic as more fundamental than geometry, and had he possessed a well-developed conception of the set-membership relation, he might have given a very different account of mathematical cognition. As it was, the Euclidean tradition, and in particular the Eudoxian theory of proportions, with its emphasis on concrete continuous magnitudes over discrete magnitudes, steered Kant toward a mereological account of mathematical cognition.

If my interpretation is correct, Kant's philosophy of mathematics is more eclectic than one might have expected. On the one hand, Kant engaged with some of the pressing mathematical issues of his day; inspired by the fluxional calculus and its attempt to address difficulties posed by infinitesimals, he developed a theory of cognition that was founded on a continuous figurative synthesis.[59] On the other hand, Kant directly engaged with the Greek mathematical tradition. These two influences converge in the synthesis of composition underlying our representation of continuous magnitudes.

It seems to me that Kant's eclecticism is a consequence of his desire for completeness and his respect for two millennia of philosophy. Kant does not believe that a clean slate is the best way to progress. One sees this in his attempt to reconcile empiricism and rationalism while improving on both. One also sees this in the Transcendental Dialectic, which shows far more tolerance of scholastic notions than modern readers will. In this respect, he is much more like Leibniz than, say, Hobbes. From a modern standpoint this side of the arithmetization of mathematics, one might think that Kant showed too much respect for the Greek mathematical tradition. Indeed, even in his own time, he was somewhat of a throwback in his insistence on the centrality of spatial magnitudes and geometry. Kant was not, like Leibniz, an original and contributing mathematician, nor did he aim to be. Kant's focus was philosophical: he was concerned with the epistemology of mathematics and the nature of mathematical cognition. Given his concerns and the fact that he was writing before the complete arithmetization of mathematics, it is quite natural that his theory of mathematical cognition gives such a prominent role to the cognition of magnitudes.

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Notes

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lengthy discussions and for detailed comments and discussions through the long gestation of this work; their help was invaluable. I am also indebted to two anonymous reviewers for incisive suggestions. Early versions of this paper that focused on the inadequacies of conceptual representation were presented at Indiana University, Trinity College, University of Illinois at Chicago, University of Toronto, University of Virginia and as a guest lecture in a seminar at Harvard University in 1999, and a recent version was discussed in a seminar at UIC; I would like to thank the participants for their helpful responses.

[1] Concepts can also relate to other concepts, but in human theoretical cognition, an intuition must ultimately mediate between any concept and its object.

References to the Critique of Pure Reason will be, in the standard way, to the original pagination of the first (A) and second (B) editions; all other references to Kant's work will be to volume and page number, separated by a colon, of the Akademie edition of *Kants Gesammelte Schriften*. References to lectures on metaphysics are followed by the best estimate of the date of the lecture. All translations from the German are my own, though I have closely consulted Guyer and Wood 1998, Hatfield 1997, Young 1992, and Ameriks and Naragon 1997.

[2] I have primarily in mind the work of Jakko Hintikka (1969, 1972, 1974a, 1974b), Charles Parsons (1969, 1984), Manley Thompson (1972), Robert Howell (1973), Emily Carson (1997), and Michael Friedman (1992, 2000).

[3] Kant distinguishes between two notions of magnitude, *quantum* and *quantitas*. The Latin suggests that the two are related as relatively concrete to more abstract, and this is how Kant understands them. In the Axioms of Intuition, Kant provides a definition of *quantum* and argues that all appearances *are* magnitudes in this sense (B202–3). Thus, objects like walking sticks don't simply *have* magnitudes; they *are* magnitudes. Kant's argument turns on the claim that apprehension of an appearance requires the representation of a determinate space or time, and it is in virtue of this relatively concrete determinate space or time that an appearance is a magnitude. Although I briefly discuss Kant's more abstract notion of magnitude (*quantitas*), the focus of the present paper will be on Kant's concrete notion of magnitude (*quanta*). See note 9 below.

[4] For a discussion of this point, see Stein 1990, 163–66.

[5] For examples of this view, see Walsh 1975, 110–11, and Kitcher 1982, §5.

Kant's theory of magnitudes is also obscured by his presentation of it in the *Critique of Pure Reason*. I think, for example, that there have been misunderstandings of the treatment of mathematics in the Transcendental Aesthetic on the one hand and the Transcendental Analytic on the other, which have encouraged misinterpretations of Kant's discussion of magnitudes in the Axioms of Intuition. Perhaps encouraged by the "arithmetization" of mathematics, some scholars hold that the Axioms of Intuition concern only the application of mathematics to objects of experience, while others think that the Axioms concern only the introduction of a metric into space and time (or into the measurement of objects in space and time). This has discouraged appreciation of important claims of the Axioms of Intuition that concern any mathematical cognition whatsoever. I argue for this position in Sutherland 2005a.

[6] Brittan 1978, for example, includes a thoughtful and seminal discussion; Parsons 1984 provides a careful assessment of Kant's use of different notions of magnitude in a variety of texts; Friedman 1992 includes a lucid analysis of the notions of magnitude, primarily in Kant's philosophy of arithmetic; and Longuenesse 1998 also provides a penetrating reading of the concepts of magnitude in that context.

[7] Charles Parsons (1984), for example, explores the relation between Kant's notions of quantity and magnitude and his understanding of arithmetical concepts in particular. Michael Friedman (1992, 104) provides an illuminating account of Kant's geometry, while his discussion of magnitudes is found in a more general chapter on the mathematical sciences, which focuses on arithmetic and algebra. Beatrice Longuenesse's treatment of magnitudes (1998, chap. 9) begins with number and arithmetic and only then considers the spatial magnitudes of geometry.

[8] A complete account of Kant's philosophy of mathematics would require a treatment of his more abstract notions of magnitude, and hence consideration of points brought forward by the authors mentioned in the previous note and by other authors. For more on the distinction between Kant's abstract and concrete notions of magnitude, see my Sutherland 2004. I discuss Kant's views on arithmetic and algebra in more detail in Sutherland, Forthcoming.

[9] For a discussion of this role and its recognition, see Sutherland 2005a.

[10] Kant also allows for this role at A162/B202, where he says that the principles of mathematics all acquire their possibility from the principles of the Axioms and Anticipations.

One might counter that Kant thinks that demonstrating the real possibility of mathematics amounts to establishing its application to the objects of experience and hence does not concern pure mathematics after all. Kant's description of the Axioms and Anticipations as concerning the application of mathematics to experience encourages this view. I think, however, that in Kant's view, the conditions for pure mathematics and the conditions for its application are much more closely related than this interpretation suggests. Longuenesse makes essentially the same point in her careful analysis of *synthesis speciosa* (1998, 270 and 274–75). I return to this issue in note 39 below; for a more extended discussion, see "The Point of Kant's Axioms of Intuition."

[11] The two-step structure of the argument of the B-edition Axioms of Intuition—first for magnitudes, then for extensive magnitudes—has not been recognized; I also think that the definition has not been properly understood. For a full discussion of the structure of the argument and the definition of magnitude, see Sutherland 2004.

[12] There is much careful philosophical scholarship attempting to reconstruct Eudoxian and pre-Eudoxian theories of proportion. This work, however, does not bear on the present paper and I will refer to the Greek theory of proportions as Euclid's; the basic features of the theory presented in Euclid and passed down through the Greek mathematical tradition are sufficient for our purposes.

[13] References to Euclid's statement of definitions, axioms, and propositions will be to Ian Mueller's translation (see Mueller 1981, 317–70). All other translations of Euclid are from Heath (see Euclid 1956).

[14] It is not entirely clear what kinds of things the Greeks counted as magnitudes. Although the kinds mentioned were clearly paradigmatic, angles, weights, and other things may have been counted as magnitudes as well. Aristotle, who followed Eudoxus and preceded Euclid, does not use the term 'magnitude' when discussing ratios and proportions, and he may have wished to reserve the term for geometrical magnitudes; see Mueller 1970, 1. Euclid, on the other hand, thought of numbers as magnitudes, as I discuss below in note 16.

[15] The requirement of homogeneity for standing in ratios may not have been a part of Eudoxus' original theory, but was a requirement by the time of Euclid (see Mueller 1970). Two inhomogeneous magnitudes could be called "incommensurable," meaning that two magnitudes cannot be compared at all, but this should not be confused with the notion of incommensurability discussed below, which means that two magnitudes have no common measure. Two magnitudes could be comparable without having a common measure; for example, $\sqrt{2}$ is smaller than 2 and hence comparable, but it shares no common measure with 2.

[16] See Stein 1990 for a lucid discussion of the theory of ratios to which I am indebted. This roundabout way to define sameness of ratio avoids the problem of incommensurables. The scope of the present paper unfortunately prevents a more complete account, which belongs to a paper on Kant's treatment of discrete magnitudes. Nonetheless, a brief description of the problem and its solution motivates the priority of geometry over arithmetic, helps explain the relation between books 5 and 7, and reveals further influences of the Greek mathematical tradition on Kant.

Euclid defines a number as a multitude composed of units (bk. 7, def. 2). As in the case of magnitudes more generally, there was a tendency in Greek thought to think of numbers concretely; numbers were thought of as multitudes of particular things; for example, the two of my eyes would be distinct from the two of your eyes (see Stein 1990, 163–66). The ontological status of numbers in Greek thought is a complex issue, but if we set this aside there is another extremely important difference from the modern conception. Because the Greeks thought of numbers as multiples of units, they held that there are only numbers corresponding to the positive integers; according to the Greeks, there simply are no numbers corresponding to what we call the rational numbers, such as $\frac{2}{3}$. This did not limit their mathematics, however, since the theory of ratios and proportions allowed them to treat the ratios between natural numbers, that is, by using the ratio 2:3.

There are, however, ratios between magnitudes that cannot be expressed as the ratio of two natural numbers. The ratio between the diagonal and the side of a square, for example, cannot be expressed as such a ratio because there simply is no unit, no matter how small, that will divide both the diagonal and the side of a square a whole number of times. In other words, the two magnitudes are "incommensurable." Thus, in the Greek account, there is a more restricted range of ratios in which numbers can stand than magnitudes more generally.

Given the Greek conception of number, Euclid had good reason not to consolidate the accounts of magnitude in books 5 and 7. Book 5 assumes the existence of a fourth proportional for any three; expressed algebraically it assumes that for every homogeneous magnitude pair a, b, and another magnitude c, there is an x homogeneous with c such that $a{:}b = c{:}x$. This condition is not fulfilled for numbers; for example, there is no whole number corresponding to $2{:}5 = 1{:}x$. (I would like to thank Bill Tait for pointing this out to me.) Despite their separate treatment, numbers are still magnitudes: recall that a ratio is a relation in respect of size between two magnitudes of the same kind and that numbers can stand in ratios. I would like to thank Lisa Shabel for prompting me to make this point explicit.

The restricted range of numerical ratios indicated a deficiency or incompleteness to the Greeks and entailed that numbers could not be used to describe all mathematical relations. These factors led the Greeks to give priority to spatial magnitudes over numbers and to geometry over arithmetic, a persisting legacy of the Greek mathematical tradition quite out of spirit with the eventual arithmetization of mathematics. It strongly influenced many mathematicians and philosophers through the early modern period, including Kant.

Kant held that the complete concept of a number requires a determinate relation to unity, a condition that rational numbers satisfied since the numerator and denominator each satisfied it. The ratios between incommensurable magnitudes did not, however. Such a ratio can at best be numerically approximated, as 1.141592 only approximates $\sqrt{2}$. Kant's view ultimately rests on the claim that a complete concept requires a completion or totality in the determination of the concept. He thinks this implies that the ratio between the diagonal of a square and one of its sides cannot be expressed by numbers, and holds that geometry can represent magnitudes that arithmetic cannot (11:207–10). His conclusions agree with the Greek view. As in other cases discussed in this paper, Kant's approach to mathematics is through mathematical cognition, and he finds features of mathematical cognition that explain features of mathematics.

I agree with Michael Friedman that in Kant's view, universal arithmetic, that is, algebra, goes beyond the arithmetic of numbers by allowing the representation of incommensurable magnitudes, and that the arithmetic of numbers corresponds approximately to book 7 while algebra corresponds approximately to book 5. See Friedman 1992, 109–21 for a careful and helpful discussion of these difficult matters. I do not agree, however, with his suggestion that algebra differs from arithmetic of numbers in adding the concept of ratio (see Friedman 1992, nn. 31 and 32). The arithmetic of numbers developed in book 7 also presupposes the notion of ratio. For a more detailed discussion of Kant's arithmetic and its relation to algebra, see Shabel 1998 and Sutherland, Forthcoming.

[17] I owe this algebraic form to Heath (Euclid 1956, 2:139 and 170), though I have simplified the first case by considering just three terms. Heath in turn is quoting De Morgan's entries for *Ratio* and *Proportion* in *Penny Cyclopedia*, vol.

19, 1841. DeMorgan states that the first six propositions are simple propositions of "concrete" arithmetic.

[18] There are also laws governing magnitudes that apply to particular kinds of magnitudes, for example, the propositions of geometry that depend on specifically geometrical definitions and axioms.

[19] See, for example, A734/B762, A717/B745, *Enquiry* (2:278), and the May 19, 1789 letter to Rheinhold (11:42). Kant was not alone in holding this view; Newton and Euler, for example, shared it. See Sutherland, Forthcoming, for a more thorough treatment of Kant's views on arithmetic and algebra.

[20] I discuss the early modern understanding of the theory of proportions in more detail in Sutherland, Forthcoming.

[21] As I stated in the Introduction, I will not attempt a detailed historical reconstruction of the reception of the Greek mathematical tradition in Kant's time, but I would like to make a few comments on the availability of the Euclidian theory of proportion. Many editions of Euclid in Kant's day altered and abbreviated the proofs, and many included only books 1–6, or books 1–6 and books 11 and 12; that is, they included the books of plane geometry or plane and solid geometry, skipping the arithmetical books 7–9 and book 10 (on incommensurables). The Latin editions by Andre Tacquet (1654) and by Claude François Milliet Dechales (1660), among the most popular, only included the eight geometrical books, at least in their earliest editions (Heath 1956, 1:105–6). Even for those without access to books 7–9, however, book 5 would have provided a good understanding of the theory of proportion. Furthermore, Tacquet also published *Arithmeticae theoria et praxis* (1656), which included the arithmetical books (Wolff 1750, §13), and an improved edition of Dechales' *Mundum Mathematicum* appeared in 1690, which included all the books of Euclid (Wolff 1750, §263). Both works went into multiple editions (see the *Dictionary of Scientific Biography*, "Tacquet," 13:236, and "Dechales," 2:622). There were other good Latin translations of the entire *Elements* in 1572, 1655, and 1756, and German translations in 1714 and 1781. There was a German translation of just the arithmetical books in 1558; see Heath 1956, 1:104–8.

Kant's exact sources for Euclid are uncertain. Kant kept a relatively small library (about 450 books), and there are difficulties with using it as an indication of his sources: it may at one time have included more, Kant certainly borrowed and read many books that were not in his library, and, especially toward the end of his life, Kant received unrequested books from authors and publishers. We can be relatively sure that his library contained at least twenty-eight particular books on arithmetic, geometry, trigonometry, universal and elementary mathematics, and the foundations of mathematics, but the list includes no editions of Euclid (see Warda 1922, 7–16 and 38–40).

Notes on Kant's mathematics lectures in 1762–64 provide an important clue to the availability of the arithmetic books and Kant's own knowledge of them. Kant states, "Already 2000 years ago Euclid quite demonstratively explained the properties of numbers in Books VII – IX, which in newer editions have been grievously omitted. Taquet [sic] translated them" (29:52). Kant is referring either to a late edition of Tacquet's *Elementis Planae et Solidae*

(1701), which by that late edition may have included the arithmetical books (as suggested by 29:683fn52$_{11}$), or to his *Arithmeticae theoria et praxis* mentioned above. Although omitted in many if not most editions of Euclid, the arithmetical books were available, and Kant thought highly enough of them to think their omission grievous.

[22] In this interpretation of Kant, not only geometry but the arithmetic of numbers and algebra concern magnitudes, which are homogeneous manifolds in intuition. I think that Charles Parsons, in his seminal article "Kant's Philosophy of Arithmetic" (1969), overlooks the full significance of the difference between "a concept of a thing in general by determination of magnitude" and "a concept of a thing in general." Although he remains concerned about the intuitive conditions for the determination, I think he misses the role of Kant's conception of magnitudes (134–35).

Friedman (1992, 112–22) offers an interpretation of Kant's philosophy of mathematics in which the arithmetic of numbers and algebra "provide us with the concept of a thing in general" (113), and "do not assume anything specific about the nature and existence of the object of our intuition" (113); he also states that "in an important sense, arithmetic does not concern objects of intuition at all" (122). This position is connected to his view that Kant conceived algebra and arithmetic as "techniques of calculation for solving particular problems, for finding the magnitudes of any objects there happen to be—where the latter are not given by the sciences of arithmetic and algebra themselves." Time is a condition of these calculations in the successive addition of unit to unit (116), but arithmetic and algebra completely abstract from the nature of the objects treated.

Friedman adds in a footnote that, more carefully stated, arithmetic and algebra provide us "with the concept of *an object of intuition in general*" (1992, 114 n. 34), and he cites texts that refer to the synthesis of the manifold homogeneous in intuition in general and the concept of magnitude. His discussion of arithmetic and algebra also makes clear that they concern, as Kant states, the "concept of a thing in general through the determination of magnitude." Nonetheless, Friedman's interpretation does not emphasize the fact that arithmetic and algebra concern magnitudes and presuppose their mathematical homogeneity.

This is not to take issue with the roles of intuition for which Parsons or Friedman are arguing—for example, that in Kant's view the intuition of time either provides a model for or is necessary for the very possibility of representing iteration (Parsons 1984, 116; Friedman 1992, 116-22). It is rather that important features of Kant's philosophy of mathematics will be missed if we overlook the role of magnitudes. (It should also be noted that Friedman later extends his interpretation in interesting and important ways in "Geometry, Construction and Intuition in Kant and his Successors" (2000). He does not, however, significantly alter his treatment of magnitudes.)

In my view, arithmetic and algebra are more abstract than geometry, and Kant's account of number focuses on the rule or schema for magnitudes rather than on the objects considered. Nonetheless, arithmetic and algebra cannot abstract from the fact that the objects concerned are magnitudes, that is, homogeneous manifolds in intuition in general. It is this fact that makes arithmetic and algebra possible at all.

[23] This notion of homogeneity reflects its etymology, as it derives, through Latin, from the Greek for 'same genus'. The German, 'gleichartig', derives from 'same species'.

[24] Longuenesse holds that this is the notion of homogeneity Kant has in mind for the homogeneity of magnitudes (1998, 249–50, esp. n. 16). As will become apparent, I think that Kant appeals to a strict notion of logical homogeneity that requires intuition for its representation.

[25] See note 8 above. The focus in this paper on *quanta*, and hence concrete magnitudes, reflects the aim of determining the role of intuition. A more complete account of the role of the categories and schemata in Kant's philosophy of mathematics would require an independent treatment of *quantitas*. I discuss the distinction between *quanta* and *quantitas* in more detail in Sutherland 2004.

[26] I follow the editors of Kant's Lectures on Metaphysics in thinking that the term 'latter' had been mistakenly inserted for 'former' either by Kant, by the student taking the notes, or by a transcriber (students very often hired someone to rewrite their notes). See Kant 1997, 460, note b. The last line of the quote substantiates the claim that this mistake had been made.

[27] In an extensive survey of passages, I have not been able to detect any distinction between Kant's use of 'gleichartig' and 'homogen', and I take them to be synonymous.

[28] There is an apparent difficulty with Kant's account of bare numerical difference, which he describes as multiple instances of an *infima species*. Kant claims in the *Critique* that it is a presupposition of the faculty of reason that there are no *infimae species*. This presupposition is based on the fact that concepts are general representations, that is, they are capable of referring to more than one thing by means of a common characteristic (A655–56/B683–84). If this were true, however, then following the line of thought sketched above, there could be no representation of strict logical homogeneity or of magnitudes. But if Kant does not think that there are *infimae species*, why does he explain the maximal logical homogeneity of magnitudes in reference to them?

The solution of this difficulty is found in two ways of considering concepts. Kant claims that if we consider a concept apart from all relation to an object, that is, consider it from the point of view of logic, we see that because it is a general representation, it is always possible in principle to add further specifications to it. For that reason, a concept is by its very nature not an *infima species*. The claim that there are no *infimae species* is a principle of logic.

According to Kant, however, we can also consider concepts as they are employed in the cognition of objects. Such concepts are not qualified *ad infinitum*. In the *Jäsche Logik*, Kant states that when we employ a concept in the cognition of an individual, we either do not notice or ignore the further possible specifications of the concept: "Only *comparatively for use* are there lowest concepts, which has attained this meaning, as it were, through convention, insofar as one has agreed not to go deeper here" (9:97) and "A *species infima* is only comparatively *infima*, and is the last in use. It must always be possible to find another *species*, whereby this latter would in turn become a *genus*. But applied immediately to *individua*, a species can be called a *species infima*" (24:911).

[29] I would like to thank Tyler Burge for assistance in clarifying several points in this argument.

This limitation of conceptual representation requires two clarifications. First, Kant thinks that we possess concepts that are derived from intuitions; the concept of a region of space, for example, is based on an intuition of space. Such concepts draw on the properties of intuition for their content and may not be subject to the same representational limitation. Thus, we might derive a concept of bare numerical difference from intuition, but we cannot represent that difference by means of concepts on their own. Second, even concepts on their own could allow us to characterize a strict logical homogeneity negatively as a difference where there is no specific difference. (I would like to thank Robert M. Adams for prompting me to consider this possibility.) This negative characterization, however, does not yield a representation with any positive content, and does not give us the ability to conceptually represent bare numerical difference by means of concepts.

[30] See in particular notes 43 and 52 below.

[31] There are two notions of numerical diversity in play here: the numerical diversity of discrete objects and the continuous numerical diversity of regions of space, which Kant describes as the "plurality and numerical difference given to us by space itself." Kant argues that intuition allows us to represent the numerical diversity of two water drops, that is, the numerical difference of discrete objects. He argues that if we abstract from any difference of quantity, and hence any difference between the amount of space each object occupies, their simultaneous position in different locations in space would still distinguish them. In other words, the discrete numerical diversity of indistinguishable objects is grounded in the continuous numerical diversity given by intuition.

[32] Keeping in mind, as we noted earlier, that the algebraic notation is a modern anachronism used to clarify a proposition that concerns the composition of magnitudes in general.

[33] I would like to thank Bill Hart for pointing out the interdependence of homogeneity and composition in my interpretation.

[34] The idea that only homogeneous magnitudes can be composed carried over into the understanding of algebra found in the Euclidean tradition. Algebraic terms were deemed meaningful only insofar as they stood for geometrical quantities. Thus, algebraic terms of different powers were thought to correspond to the different homogeneous kinds of elements: terms of first power to lines, second power to areas, and third power to volumes. The words 'squared' and 'cubed' reflect this manner of thinking of algebraic terms. The Euclidean understanding of algebra influenced renaissance and early modern developments in algebra. Some early modern mathematicians thought of algebra as primarily a method of solving geometrical problems, and solutions obtained algebraically had to be geometrically constructed in order to count as proper proofs. See Shabel 1998 for a careful assessment of this understanding of algebra and Kant's own views. Also see my "Kant on Arithmetic, Algebra,

and the Theory of Proportions."

[35] Although this appears a reasonable conclusion, it is one that mathematicians would reject today. The real line can be thought of as a set of continuum-many points, each without extension. Similarly, a real plane can be thought of as a set of continuum-many lines, each without width. The modern arithmetized approach to geometry and the notion of set-membership relation upon which it is based allows us to circumvent the paradox.

[36] This common notion may have been added to the *Elements* after Euclid but was generally accepted. Two narrower notions of part or parts appear in the definitions of books 5 and 7, but a part in either of these senses is also smaller than the whole, in conformity with common notion 5; see Euclid 1956, 1: 232, 2:113–15 and 277–80.

Common notion 5 entails that a magnitude cannot have the same size as a proper part of it. From a modern standpoint, an infinite magnitude can have the same size as a proper part of it. We have already set aside infinite magnitudes, however, since we are considering only those that conform to the Archimedean property and hence can stand in ratios to one another. This is not to say that a whole the same size as a proper part was a possibility recognized and ruled out by the Greeks; nor was it seen as a possibility by Kant, who thought it analytic that a whole is greater than its part (B17). Scholastics and, famously, Galileo recognized that an infinite collection is equinumerous with a proper part and that in this sense a whole is equal to a proper part; it was thought a paradox, and in response Galileo denied that equal, greater, and less applied to infinite quantities (Galilei 1954, 31–33). Although it may have appeared in Bolzano, it probably was not until Dedekind that the possibility of a collection being equinumerous with a part was used as a criterion of the infinite (1901, 63).

[37] There is a converse relation concerning composition that is also assumed: for any part of a magnitude, there is another whose composition with the first is the whole magnitude. I would like to thank Bill Tait for pointing this out to me.

There is a difference between the way that the term 'homogeneous' is used in Euclid's *Elements* and in Kant that can now be clarified. In the *Elements*, 'homogeneous' is predicated of two or more magnitudes in respect of each other, and hence concerns the relation *between* certain magnitudes. In Kant, 'homogeneous' is predicated of the manifold contained *within* a magnitude, and hence concerns the constitution of any magnitude.

Despite this difference in the way the term 'homogeneous' is used, the underlying notion of mathematical homogeneity on which both views rest is the same. In the *Elements*, magnitudes are homogeneous with each other if and only if they belong to the same class of things that allow of composition and hence of standing in ratios. (The Greek concept of magnitude presupposes the notion of mathematical homogeneity, for to be a magnitude is to belong to some class of things that are mathematically homogeneous with each other.) Given the Greek view of composition, a magnitude and all the parts that compose it belong to the same class of mathematically homogeneous elements. This is reflected in Kant's definition, according to which something is a magnitude in virtue of the mathematically homogeneous parts that compose it. (I would like to thank Lisa Shabel and Bill Tait for raising this issue.)

[38] Requiring an ability to compose is a quite natural view of the conditions of measurement that corresponds to what is now called standard measurement. It turns out to be too restrictive. In the 1950s, the conditions of measurement were axiomatized, and it was demonstrated that measurement is possible even in cases where there is nothing corresponding to the composition relation (see Krantz, Suppes, and Tversky 1971, 1:6–7).

[39] This result allows us to make an important point concerning the Greek view and Kant's views on the application of mathematics to the world, alluded to in footnote 10 above. As mentioned in the Introduction, from a modern viewpoint, the foundation of mathematics begins with numbers, and it is a further issue whether numbers can be applied to magnitudes in order to measure them. In the Greek account, mathematics is rooted in the study of homogeneous magnitudes, and the investigation into the assumptions underlying their mathematical properties simultaneously reveals the conditions for their measurement, since the mathematical properties and the conditions for measurement rest on the same fundamental relations of composition and relative size of magnitude. Similarly, Kant holds that uncovering the conditions of the mathematical cognition of magnitudes will simultaneously reveal the conditions for the application of mathematics to objects of experience, which are all magnitudes. Longuenesse makes essentially the same point in her discussion of *synthesis speciosa* (1990, 270, 274–75), as does Friedman in his discussion of the real possibility of mathematics (1992, 93).

[40] By defining smaller and larger in this way, we are assuming that a magnitude cannot have the same size as a proper part of it, as mentioned in footnote 36 above. It also assumes the existence of a part of the larger magnitude equal to any smaller magnitude, whatever size the smaller magnitude may be. Modern mereology articulates these and other assumptions. See Simons 1987.

[41] More carefully, a magnitude, in order to be either extensive or intensive, must be determinate and hence represented through the synthesis of composition, about which I will say more below. The representation of space discussed in the Transcendental Aesthetic is not determinate and is neither extensive nor intensive, although it is a magnitude. For more on this issue, see "The Point of Kant's Axioms of Intuition."

[42] Since Kant holds that space is infinitely divisible, he will also maintain that there are an unlimited number of parts in a region of space. He is not, however, claiming that we apprehend all of them as parts. Kant has in mind the fact that we cannot apprehend a line, for example, without apprehending it as having distinguishable parts, even if we do not explicitly cognize or otherwise pick out all the possibly distinguishable parts.

[43] Kirk Dallas Wilson (1975) has also emphasized the importance of part-whole relations in Kant's conception of intuition. My interpretation differs significantly in the way it connects those mereological properties to mathematics. It also differs in relating them to the Greek mathematical tradition.

The nature of intensive magnitudes raises an issue concerning the role of intuition. An intensive magnitude is a magnitude and is therefore a homogeneous manifold in intuition in general. This implies that an intensive magnitude is mathematically homogeneous, that is, that it allows composition relations between smaller and larger intensities conceived as parts and wholes. Furthermore, it implies that an intensive magnitude is a strict logical homogeneity. A particular color of light, for example, will differ in quality from other colors, and hence be specifically distinguished from them, but the different intensities of that light will be homogeneous with each other. Kant implies this in the *Prolegomena*:

> although sensation, as the quality of empirical intuition with respect to that by which a sensation is distinguished specifically from other sensations, can never be cognized *a priori*; it nonetheless can, in a possible experience in general, as the magnitude of perception, be distinguished intensively from every other homogeneous sensation. (4:309)

He makes the same point in the First Introduction to the *Critique of Judgment*: "One can definitely say: that things must never be held to be specifically different through a quality that passes into some other through the mere diminution or augmentation of its degree" (20:226 n.). See also the Metaphysik Vigilantius, where Kant refers to the "manifold homogeneous" of an intensive magnitude (29:1000, 1794–95).

Since sensations can represent a strict logical homogeneous magnitude, it might seem that intuition is not required to represent it after all, contrary to the argument outlined in part 1 above. Kant makes it quite clear, however, that if it were not for intuition, and time in particular, we could not even represent the intensive magnitude as containing a manifold. Thus, intuition is required in order for us to *directly* represent extensive magnitude; it also allows us to *indirectly* represent an intensive magnitude as a magnitude by means of directly representing extensive magnitude. This is what Kant means in the *Prolegomena* when he states that the principle of the Anticipations

> does not subsume … sensation … directly under the concept of *magnitude*, as sensation is no intuition *containing* space or time, although it places the object corresponding to it in both. (4:306).

When Kant defines a magnitude as a homogeneous manifold *in intuition*, the sense of 'in' comprises both intuitions that contain space and time, and sensations whose objects are placed in space and time.

44 Parsons has noted (1984, 112) that part-whole notions dominate in Kant's explanation of the categories of quantity. The development of Kant's views is more obscure and complex than my summary suggests; the connection between unity-plurality-totality, one-many-one, and part-whole is mediated by Kant's views on unity, truth, and perfection. The evidence is spread throughout Kant's notes in his copy of Baumgarten's *Metaphysics*, and his own notes on metaphysics (1902, vols. 17 and 18). Filling out the details, however, would take us too far afield.

45 In a letter of 1797, Kant even refers to the category of intensive magnitudes:

> All the categories are directed upon some material composed a priori; if this material is homogeneous, they express mathematical functions, and if it is not homogeneous, they express dynamic functions. Extensive magnitude is a function of the first sort, for example, a one in many. Another example of a mathematical function is the category of quality or intensive magnitude, a many in

one. (12:222–25)

[46] This interpretation differs from that of Parsons (1984). As discussed in footnote 7 above, Parsons is primarily interested in Kant's understanding of arithmetical concepts and their relation to the categories of quantity. This leads him to focus on those texts that indicate a set-element relation rather than the part-whole relation. He states: "Kant does not distinguish very clearly between the whole/part and the set/element relation. I will show, however, that there is some basis, even though not clearly articulated, for Kant to make the distinction. Something like the latter relation is needed to make sense of the relation of the categories to the concept of number" (1984, 113). In my view, it is reasonably clear that the application of the categories of quantity to intuition concerns the part-whole relation.

This interpretation also differs from that suggested by Longuenesse (1998). I cannot do justice here to her account of the relation between the quantitative forms of judgment and the categories of quantity, so I limit myself to pointing out that her analysis focuses on the concept of number and arithmetic and hence on Kant's more abstract conception of discrete magnitude (*quantitas*) rather than concrete magnitude (*quantum*). In her view, only *quantitas* "is strictly speaking an instance of the category of quantity," and Kant's references to the categories or category of magnitude should be understood as the category of *quantitas* (1998, 266). The schema of *quantitas* is number, so that her account of the role of the categories focuses on discrete magnitudes—in particular, on discrete objects that fall in the extensions of concepts (257). The categories of *quantitas* are used to determine the *quantitas* of a *quantum* (that is, its size), but there appears to be no role for them in the cognition of the part-whole relations of *quanta*. In my view, the categories can be employed in the generation of our representations of *quanta* and our cognition of their part-whole relations, prior to and independently of their employment for measurement.

As noted earlier, a complete account of Kant's philosophy of mathematics will have to consider Kant's philosophy of arithmetic as well, for which, see Sutherland, Forthcoming.

[47] See, for example, 11:208; 29:994, 1794–95. For an example involving arithmetic, see 2:397.

[48] As has been shown by recent work, Kant's engagement with this issue shaped his account of mathematical cognition: influenced by the Newtonian geometric-kinematic interpretation of the calculus, Kant posited a continuous figurative synthesis that generates geometrical representations through the movement of a point in space. See Kitcher (1975), and see especially Friedman (1992, 55–95), who provides a penetrating analysis and exposition of the kinematic conception, its limitations, and its influence on Kant.

[49] Kant sometimes uses 'composition' in a broader sense for any sort of combination; it is the more narrowly defined sense of composition that interests us here. See also *The Critique of Practical Reason*, 5:104.

[50] This synthesis of composition is also the same figurative synthesis underlying Kant's kinematic conception of mathematics; see footnote 48 above.

Kant's cognitive account of composition as an act of synthesis lends itself to obscuring the distinction between the composition relation in which magnitudes stand and the operation of composing magnitudes.

[51] In her discussion of Kant's arithmetic, Longuenesse draws a similar contrast between the combination of marks in a concept and the generation of a multiplicity from two given multiplicities "by means of an operation which has nothing in common with the combination of marks making up the content of a concept" (1998, 277). The conclusion she draws is quite different, however; in her view, this shows that Kant's concept of number is implicitly a second-order concept, one that reflects a rule for constituting the extensions of concepts. In my view, the contrast shows that mathematics requires a special synthesis of combination of a strictly logically homogeneous manifold, that is, intuition.

[52] It may appear that strict logical homogeneity is too stringent a requirement for mathematical composition, which is surely not restricted to qualitatively identical parts of space and time. I would like to respond to three different ways this issue might be raised. First, one might worry that a combination of blue and red blocks of the same size, say, into larger groupings would not count as mathematical composition, because the blocks are qualitatively distinct. This construes the role of strict homogeneity too narrowly, however. There are different features of the blocks, and the feature that is mathematically composed, their spatial extension, is strictly logically homogeneous. Colors can be mathematically combined under the right circumstances; that is, when the colors are strictly homogeneous and when they are combined in such a way that their intensities can combine. (I will say more about intensive magnitudes below.) In the present case, however, their colors are not combined in a way that could count as mathematical composition. Stated in another way: what must be strictly homogeneous are the properties that are mathematically composed, not all the properties of the combined objects.

Second, one might worry that requiring qualitative identity of the manifold is too stringent even in the paradigm case of spaces, since we can mathematically combine spaces of different sizes. Lines of different lengths, for example, differ in quantity, yet they can be mathematically composed into a longer line. This worry is reinforced by the fact that Kant's discussion in the Amphiboly refers to parts of space that are identical in both quality and quantity. Here it must be said that although the Amphiboly has important implications for mathematical cognition, mathematical cognition is not its focus. Kant is arguing against Leibniz's assumption that all representation is fundamentally conceptual in nature, and he does so by pressing on Leibniz's principle of the identity of indiscernibles. His argument focuses on the case in which the spaces or the rain drops are identical in both qualitative properties and quantitative properties such as size, but this is not a requirement for all mathematical composition. He includes these quantitative properties for the sake of his argument against Leibniz. More importantly, Kant does not need to distinguish here between two notions of magnitude, *quanta* and *quantitas*. The former is a strict logical homogeneity, while the latter comprises the determinate quantitative properties of a *quantum*. While it is true that spaces that are identical in both quality and quantity are strictly logically homogeneous, this is not a requirement of strict logical homogeneity. What is required is that there not be any qualitative difference in the manifold making up the *quantum*, and lines of different sizes meet this requirement. For more on the distinction between *quantum* and *quantitas*, see my "The Role of Magnitude in Kant's Critical Philosophy." For a discussion of Kant's understanding of inner determinations, outer relations, quality, and quantity, and their relation to *quantum* and *quantitas*, see Sutherland 2005b.

The third worry is that if only space and time can be strictly logically homogeneous, then only spaces and times are magnitudes. Kant, however, clearly thinks of intensive magnitudes, such as the intensity of a light, as magnitudes. The answer to this worry denies the antecedent of the conditional: space and time are paradigm cases of strict logical homogeneity, but Kant holds that intensive magnitudes are also instances of strict logical homogeneity (see footnote 43 above for a clarification of this point). At the same time, we have seen that in Kant's view, the representation of a strict logical homogeneity requires intuition, and cannot be represented by means of concepts alone (see footnote 30 above). The homogeneous manifold of intensive magnitudes can be represented only with the aid of the extensive magnitudes of space and time, as explained in section 6. Hence, the mathematical composition of these intensities would be entirely lost to us without the aid of the extensive magnitudes of space and time. (I would like to thank Michael Friedman for pressing me to clarify Kant's position on all three of these issues.)

[53] For a discussion of Kant's views on this principle, see my Forthcoming (c).

[54] In the language of the theory of measurement, the part-whole ordering relation allows at best the creation of an ordinal scale and allows ordinal but not extensive measurement. (Krantz 1971, chaps. 1 and 3).

[55] Kant makes the same point in his discussion of magnitudes at 28:424, 1784–85; 28:561, 1790–91; and 28:637, 1792–93.

[56] The magnitudes related by size will also have to be Archimedean in the sense described on page 162 above.

Strictly speaking, the greater-than and less-than relations would not be a total ordering on the set of magnitudes, since some magnitudes will be equal to one another and hence neither greater nor less than each other. This is merely a technicality, however; we can define the relation greater-than-or-equal, which will not be a strict ordering but will be a total ordering. Alternatively, we could define equivalence classes of magnitudes of equal size and then order these equivalence classes using the greater-than or less-than relations.

[57] I have argued that in Kant's view, the categories of unity, plurality, and totality function to allow us to cognize the part-whole composition relations of intuition. The equality relation would seem to warrant a comparably important position in Kant's account of human cognition, yet no category corresponds to it. Whence the concept of equality?

I cannot give a full account of Kant's notion of equality here, but I believe that it derives from the concepts of reflection described in the Amphiboly. Specifically, it derives from the concepts of identity and diversity applied in the comparison of spatial appearances, where we understand equality to be identity of quantity and quantity to be *quantitas*, that is, the quantitative determinations of a *quantum*. See Sutherland 2005b.

[58] First, just as space is infinitely divisible into further spaces, concept extensions are infinitely (logically) divisible into extensions distinguished by further concepts (A655–56/B683–84). (This is another way of stating his doctrine that in logic there are no *infimae species* of concepts; see footnote 28 above.) Second, points are to lines as individuals are to the extension or 'horizon' of concepts; that is, a point is a limit of the extension of a line just as an individual is the limit of the extension of successive concepts in a hierarchy of concepts (A658–59/B686–87).

[59] See note 48 above.