

Class 14 - September 28
Rules of Replacement II (§7.4)

There are five more rules of replacement.

I. Transposition (Trans)

$$\alpha \supset \beta :: \sim\beta \supset \sim\alpha$$

You may switch the antecedent and consequent of a conditional statement, as long as you negate (or un-negate) both.

Often used with HS.

A statement and its transposition are traditionally called contrapositives of each other.

Sample Derivation:

1. $A \supset B$
2. $D \supset \sim B$ / $A \supset \sim D$
3. $\sim\sim B \supset \sim D$ 2, Trans
4. $A \supset \sim D$ 1, 3, DN, HS

QED

Transposition can be tricky when only one term is negated:

- $A \supset \sim B$ becomes, by Trans:
 $\sim\sim B \supset \sim A$ which becomes, by DN
 $B \supset \sim A$

Equivalently, but doing the double negation first:

- $A \supset \sim B$ becomes, by DN:
 $\sim\sim A \supset \sim B$ becomes, by Trans:
 $B \supset \sim A$

Either way, you can include the DN on the line with Trans.

II. Material Implication (Impl)

$$\alpha \supset \beta :: \sim\alpha \vee \beta$$

Implication allows you to change a statement from a disjunction to a conditional, or vice versa.

It is often easier to work with disjunctions.

You can use DM to get conjunctions.

You may be able to use distribution, which doesn't apply to conditionals.

On the other hand, sometimes, you just want to work with conditionals.

You can use HS and MP.

Proofs are overdetermined by our system - there are many ways to do them.

Sample Derivation:

1. $G \supset \sim E$
2. $E \vee F$ / $G \supset F$
3. $\sim \sim E \vee F$ 2, DN
4. $\sim E \supset F$ 3, Impl
5. $G \supset F$ 1, 4, HS

QED

III. Material Equivalence (Equiv)

$$\alpha \equiv \beta :: (\alpha \supset \beta) \cdot (\beta \supset \alpha)$$

$$\alpha \equiv \beta :: (\alpha \cdot \beta) \vee (\sim \alpha \cdot \sim \beta)$$

Equiv is almost the only thing you can do with a biconditional.

There are two distinct versions.

If you have a biconditional in your premises, you can unpack it in either way.

If you need one in your conclusion, you can get the pieces and then use this rule.

This is easier with the first definition.

Just get $\alpha \supset \beta$

Then get $\beta \supset \alpha$

Then use Conj.

Sample Derivation

1. $\sim[(K \supset \sim H) \cdot (\sim H \supset K)]$
2. $(I \cdot J) \supset (K \equiv \sim H)$ / $\sim(I \cdot J)$
3. $\sim(K \equiv \sim H)$ 1, Equiv
4. $\sim(I \cdot J)$ 2, 3, MT

QED

IV. Exportation (Exp)

$$\alpha \supset (\beta \supset \gamma) :: (\alpha \cdot \beta) \supset \gamma$$

You can sometimes get to MP or MT using Exportation.

Sample Derivation:

1. $L \supset (M \supset N)$
2. $\sim N$ / $\sim L \vee \sim M$
3. $(L \cdot M) \supset N$ 1, Exp
4. $\sim(L \cdot M)$ 3, 2, MT
5. $\sim L \vee \sim M$ 4, DM

QED

V. Tautology (Taut)

$$\alpha :: \alpha \cdot \alpha$$

$$\alpha :: \alpha \vee \alpha$$

Tautology eliminates redundancy.

Sample Derivation:

$$\begin{array}{ll} 1. O \supset \sim O & / \sim O \\ 2. \sim O \vee \sim O & 1, \text{Impl} \\ 3. \sim O & 2, \text{Taut} \end{array}$$

QED

VI. Some more potentially helpful examples

Some of these derivations may be useful as elements of other, longer proofs. Others contain useful tricks which may come in handy in other proofs.

$$\begin{array}{ll} 1. & 1. \sim A \quad / A \supset B \\ & 2. \sim A \vee B \quad 1, \text{Add} \\ & 3. A \supset B \quad 2, \text{Impl} \end{array}$$

QED

$$\begin{array}{ll} 2. & 1. E \quad / F \supset E \\ & 2. \sim F \vee E \quad 1, \text{Add, Com} \\ & 3. F \supset E \quad 2, \text{Impl} \end{array}$$

QED

$$\begin{array}{ll} 3. & 1. G \supset (H \supset I) \quad / H \supset (G \supset I) \\ & 2. (G \cdot H) \supset I \quad 1, \text{Exp} \\ & 3. (H \cdot G) \supset I \quad 2, \text{Com} \\ & 4. H \supset (G \supset I) \quad 3, \text{Exp} \end{array}$$

QED

$$\begin{array}{ll} 4. & 1. O \supset (P \cdot Q) \quad / O \supset P \\ & 2. \sim O \vee (P \cdot Q) \quad 1, \text{Impl} \\ & 3. (\sim O \vee P) \cdot (\sim O \vee Q) \quad 2, \text{Dist} \\ & 4. \sim O \vee P \quad 3, \text{Simp} \\ & 5. O \supset P \quad 4, \text{Impl} \end{array}$$

QED

5. 1. $(R \vee S) \supset T$ / $R \supset T$
 2. $\sim(R \vee S) \vee T$ 1, Impl
 3. $(\sim R \cdot \sim S) \vee T$ 2, DM
 4. $T \vee (\sim R \cdot \sim S)$ 3, Com
 5. $(T \vee \sim R) \cdot (T \vee \sim S)$ 4, Dist
 6. $\sim R \vee T$ 5, Simp, Com
 7. $R \supset T$ 6, Impl

QED

6. 1. $W \supset X$
 2. $Y \supset X$ / $(W \vee Y) \supset X$
 3. $(W \supset X) \cdot (Y \supset X)$ 1, 2, Conj
 4. $(\sim W \vee X) \cdot (\sim Y \vee X)$ 3, Impl, Impl
 5. $(X \vee \sim W) \cdot (X \vee \sim Y)$ 4, Com, Com
 6. $X \vee (\sim W \cdot \sim Y)$ 5, Dist
 7. $(\sim W \cdot \sim Y) \vee X$ 6, Com
 8. $\sim(W \vee Y) \vee X$ 7, DM
 9. $(W \vee Y) \supset X$ 8, Impl

QED

7. 1. $(J \vee K) \supset (L \cdot M)$
 2. $\sim J \supset (N \supset \sim N)$
 3. $\sim L$ / $\sim N$
 4. $\sim L \vee \sim M$ 3, Add
 5. $\sim(L \cdot M)$ 4, DM
 6. $\sim(J \vee K)$ 1, 5, MT
 7. $\sim J \cdot \sim K$ 6, DM
 8. $\sim J$ 7, Simp
 9. $N \supset \sim N$ 2, 8, MP
 10. $\sim N \vee \sim N$ 9, Impl
 11. $\sim N$ 10, Taut

QED

VII. Exercises. Derive the conclusions of each of the following arguments using the Rules of Implication and Replacement.

1. 1. $(O \cdot P) \supset Q$
 2. O / $P \supset Q$
2. 1. $R \supset (S \cdot \sim T)$ / $\sim R \vee \sim T$
3. 1. $U \equiv W$
 2. W / U
4. 1. $(H \vee I) \supset [J \cdot (K \cdot L)]$
 2. I / $J \cdot K$

5. 1. $(L \cdot M) \supset N$
 2. $(L \supset N) \supset O$ / $M \supset O$
6. 1. $A \cdot (B \vee F)$
 2. $A \supset [B \supset (D \cdot E)]$
 3. $(A \cdot F) \supset \sim(D \vee E)$ / $D \equiv E$

Solutions may vary.

VIII. Three challenging derivations. Try them.

1. 1. $A \supset B$
 2. $B \supset D$
 3. $D \supset A$
 4. $A \supset \sim D$ / $\sim A \cdot \sim D$
2. 1. $(I \cdot E) \supset \sim F$
 2. $F \vee (G \cdot H)$
 3. $I \equiv E$ / $I \supset G$
3. 1. $(J \supset J) \supset (K \supset K)$
 2. $(K \supset L) \supset (J \supset J)$ / $K \supset K$

Solutions may vary.

IX. Appendix: Proofs of the Logical Equivalence of the Last Five Rules of Replacement

Transposition: $\alpha \supset \beta :: \sim\beta \supset \sim\alpha$

α	\supset	β
T	T	T
T	F	F
F	T	T
F	T	F

\sim	β	\supset	\sim	α
F	T	T	F	T
T	F	F	F	T
F	T	T	T	F
T	F	T	T	F

Material Implication: $\alpha \supset \beta :: \sim\alpha \vee \beta$

α	\supset	β
T	T	T
T	F	F
F	T	T
F	T	F

\sim	α	\vee	β
F	T	T	T
F	T	F	F
T	F	T	T
T	F	T	F

Material Equivalence: $\alpha \equiv \beta :: (\alpha \supset \beta) \cdot (\beta \supset \alpha)$

α	\equiv	β
T	T	T
T	F	F
F	F	T
F	T	F

$(\alpha \supset \beta)$	\cdot	$(\beta \supset \alpha)$
T	T	T
T	F	F
F	T	F
F	T	F

Material Equivalence: $\alpha \equiv \beta :: (\alpha \cdot \beta) \vee (\sim\alpha \cdot \sim\beta)$

α	\equiv	β
T	T	T
T	F	F
F	F	T
F	T	F

$(\alpha \cdot \beta)$	\vee	$(\sim\alpha \cdot \sim\beta)$
T	T	F
T	F	F
F	F	T
F	F	T

Exportation: $(\alpha \cdot \beta) \supset \gamma :: \alpha \supset (\beta \supset \gamma)$

$(\alpha$	\cdot	$\beta)$	\supset	γ
T	T	T	T	T
T	T	T	F	F
T	F	F	T	T
T	F	F	T	F
F	F	T	T	T
F	F	T	T	F
F	F	F	T	T
F	F	F	T	F

α	\supset	$(\beta$	\supset	$\gamma)$
T	T	T	T	T
T	F	T	F	F
T	T	F	T	T
T	T	F	T	F
F	T	T	T	T
F	T	T	F	F
F	T	F	T	T
F	T	F	T	F

Tautology: $\alpha :: \alpha \vee \alpha$

α	\vee	α
T	T	T
F	F	F

Tautology: $\alpha :: \alpha \cdot \alpha$

α	\cdot	α
T	T	T
F	F	F