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Edited by

Dale Jacquette

Leslie Tharp
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Which Logic is the Right Logic?

Leslie H. Tharp

Introduction

It has been generally accepted in the philosophy of mathematics that elementary logic (*EL*), also known as *the predicate calculus*, or *first order logic*, yields a stable and distinguished body of truths, those which are instances of its valid formulas. I am concerned with presenting and examining evidence relevant to such a claim. In sentential logic there is a simple proof that all truth functions, of any number of arguments, are definable from (say) "not" and "and". Thus one has not overlooked any truth-functional connectives, even though one started with the few which naturally presented themselves. Operators such as "for infinitely many x", or for an arbitrary cardinal $\aleph$, "for at least $\aleph$ x", are in some ways analogous to the standard quantifier "for at least one x". If these operators are counted as quantifiers, there are many more such quantifiers than there are formulas of elementary logic; so on rather trivial grounds, there can be no theorem that all possible quantifiers are already definable in *EL*. This observation does not, however, rule out the possibility that there might be a narrower notion of quantifier for which such a theorem holds. If so, and to the extent that the narrower notion is significant, one will have evidence that *EL* is no arbitrary stopping point. I will argue that natural and satisfying criteria are suggested by the standard quantifiers which characterize arbitrary formulas of elementary monadic logic. The full logic of relations, however, appears to be more problematical.

Leslie H. Tharp, Which Logic is the Right Logic? *Synthese*, 31 (1975): 1–21.

Background

Whether or not it is a satisfactory picture of interpreted language, we shall retain the analysis which resolves interpreted language into a semantical part (models), a syntactical part (formulas), and the relation of a model satisfying (or being a model of) a formula. We do not contemplate altering the notion of model as it is used in elementary logic. A model still consists of a universe and a finite sequence of relations over the universe (for simplicity we shall ignore constants and functions). What we do contemplate is enriching the set of formulas, and thereby of course extending the satisfaction relation. Let us suppose that the set of formulas of a possible logic is a countably infinite set, and that each formula contains only finitely many letters. Restrictions of effectiveness will be introduced when they are relevant. It hardly needs to be argued that these are reasonable conditions to impose on any potential competitor of elementary logic.

As a concrete example, let us take elementary logic and define a new logic $L(I)$ by adding the symbol (Ix) which is read "for infinitely many x". It is clear how to define the resulting set of formulas, and the corresponding satisfaction relation. This logic extends *EL* in a definite sense, since the class of models satisfying $\neg (Ix)(x = x)$ is just the class of all models with finite universes, which cannot be defined by a formula of EL. (We are taking EL and other logics to contain identity.) We can also characterize the natural numbers with a formula of $L(I)$, since we can say that each number has finitely many predecessors.

These examples suggest a general framework for comparing logics. Without specifying the inner workings of a logic L, we may take it to be a collection of L-classes; each L-class may be thought of as a class of models of the form $\{M: M\mathrm{Sat}_L\ \mathscr{A}$ where $\mathscr{A}$ is a (closed) formula of L and Sat_L is the satisfaction relation for L.[1] Then we may say that a logic L_1 is contained in L_2 if every L_1-class is an L_2-class. The relations of equivalence and (proper) extension are then defined in the obvious way from containment. In the previous example EL is contained in $L(I)$, but they are not equivalent, since a certain $L(I)$-class is not an EL-class.

Within this framework one can define, for a given logic L, a notion of logical implication between a set of formulas X and a formula $\mathscr{A}$. X logically implies $\mathscr{A}$ (X1.i. $\mathscr{A}$, for short) in case all models satisfying all members of X satisfy $\mathscr{A}$. This is the concept, going back to Bolzano, which is dealt with by Tarski in "On the Concept of Logical Consequence",[2] and which we take to be a satisfactory formulation of the notion of a formula $\mathscr{A}$ following from X on "purely logical grounds". It should be remarked that our overall concern is basically the problem discussed by Tarski towards the end of his paper: Is there a sharp division of terms into logical and extra-logical?

Elementary logic is *axiomatizable*. That is, there is a proof procedure $\vdash$ such that X 1.i. $\mathscr{A}$ if and only if $X \vdash \mathscr{A}$. The proof procedure is such that a proof can involve only finitely many formulas, so if $X \vdash \mathscr{A}$ then $\Gamma \vdash \mathscr{A}$ of X. That $\vdash$ provides an axiomatization is equivalent to the conjunction of two conditions which are frequently singled out. The first is that $\vdash$ is *complete* in the sense that the formulas which are valid (i.e. true in all models) are exactly those which are provable without hypotheses. In other words, to say $\vdash$ is complete is to say ϕ 1.i. $\mathscr{A}$ if and only if $\phi \vdash \mathscr{A}$. It follows easily from completeness that for finite sets of formulas Γ, Γ 1.i. $\mathscr{A}$ if and only if $\Gamma \vdash \mathscr{A}$. The second condition is *compactness*, which says that if every finite subset of X has a model, then X has a model. Compactness is equivalent to saying that if X 1.i. $\mathscr{A}$ then, for some finite subset Γ of X, Γ 1.i. $\mathscr{A}$. Putting completeness and compactness together, one has axiomatizability. The reader should be warned that we have chosen terminology convenient for our purposes but which is by no means universally adopted. For example, "axiomatizability" is frequently used to mean our completeness.

The notions above were defined for the logic EL, and we wish to formulate them for other logics. Compactness is a purely model-theoretic notion, and, as stated, it clearly makes sense for the most general logics. Completeness can also be defined in a fairly general setting. Identify the formulas with a recursive set of natural numbers and take completeness to mean that the valid formulas are effectively enumerable. Then define a proof procedure $\vdash$ for the logic: $\{\mathscr{B}_1 \ldots, \mathscr{B}_n\} \vdash \mathscr{A}$ in case $\mathscr{B}_1 \rightarrow (\mathscr{B}_2 \rightarrow \ldots\ldots (\mathscr{B}_n \rightarrow \mathscr{A}) \ldots)$ is in the enumeration of valid formulas; for an arbitrary set of formulas X, let $X \vdash \mathscr{A}$ in case $\Gamma \vdash \mathscr{A}$ for some finite Γ included in X. (We are assuming that the logic has effective constructions corresponding to the sentential connectives.) Again in the general case compactness and completeness give axiomatizability: X 1.i $\mathscr{A}$ if and only if $X \vdash \mathscr{A}$.

A very elegant proof of axiomatizability for EL, due to Henkin, shows that every consistent set of formulas has a model. Since in fact the model produced is countable (i.e. finite or countably infinite), one has a further corollary: If X has a model, X has a countable model. This corollary is one version of a well known and somewhat controversial theorem, the Löwenheim–Skolem theorem. As a special case, if a single formula $\mathscr{A}$ has a model, then $\mathscr{A}$ has a countable model. We take this special case[3] and say that a logic L has the Löwenheim–Skolem property if every nonempty L-class has a countable member.

This leads us to the first basic technical result, an intrinsic characterization of elementary logic due to Lindström:[4] Suppose L contains EL and is either complete or compact; then if L has the Löwenheim–Skolem property, L is equivalent to EL. As an example, the logic $L(I)$ mentioned before is easily shown to have the Löwenheim-Skolem property, using the submodel proof of Tarski and Vaught.[5] Thus since $L(I)$ is not equivalent to EL, $L(I)$ is neither complete nor compact. (Of course, to show that $L(I)$ was different from EL we in effect pointed out that it was not compact.) In Lindström's theorem one does have to make a few general assumptions about the logic L. For example the L-classes must be closed under intersection and complement. This of course means that L has the sentential connectives. It is not necessary, however, to assume that L has something like quantification.[6] A few more basic restrictions are needed, e.g., that isomorphic models lie in the same L-classes. Also, for completeness one needs to bring in the obvious stipulations of effectiveness.

What Must a Logic Do?

Lindström's result gives an exceedingly sharp characterization of elementary logic as a maximal solution to certain general conditions. Elementary logic has for many years been taken as the standard logic, with little explicit justification for this role. Since it appears not to go beyond what one would call logic, the problem evidently is whether it can be extended. Thus one is tempted to look to Lindström's theorem for a virtual proof that logic must be identified with elementary logic. The success of this enterprise obviously depends on the extent to which one can justify, as necessary characteristics of logic, the hypotheses needed in the theorem, principally completeness (or compactness) and the Löwenheim–Skolem property.

A complete logic has an effective enumeration of the valid formulas. The proof procedures proposed for elementary logic were clearly *sound* in the sense that they proved only valid formulas. Gödel established completeness by showing that all valid formulas were provable by a standard proof procedure. Second-order logic, which is an extension of *EL* formed by allowing quantification over predicate letters, is a classical example of a logic with sound proof procedures, but which demonstrably admits no complete proof procedure. That is, there is no proof procedure complete with respect to the intended semantics. The standard proof procedures can be shown complete with respect to certain semantics – but such nonstandard semantics have little independent interest.

Soundness would seem to be an essential requirement of a proof procedure, since there is little point in proving formulas which may turn out false under some interpretations. But it is trivial to provide sound proof procedures: take the null procedure, or take some finite set of valid formulas. Of course the proof procedures suggested for logics such as second order logic enumerate an infinite number of valid formulas, and perhaps appear to yield, in some further sense, a large and useful set of valid formulas; in particular one may incorporate the comprehension schema which says, for each formula $\mathscr{A}(x)$, that there exists the class of those individuals x such that $\mathscr{A}(x)$.

The question is, should one demand that a logic have a complete proof procedure? In order to answer this, it seems essential to be somewhat more precise about the role logic is expected to play. One can distinguish at least two quite different senses of logic.[7] The first is, as an instrument of demonstration, and the second can perhaps be described as an instrument for the characterization of structures. In the present context, a logic L will have this latter ability if, for example, there are L-classes consisting, up to isomorphism, of a single structure of mathematical interest. Second order logic is striking in this respect. Such central theories as number theory and the theory of the real numbers seem to involve in their concepts a quantifier over all subsets of the domain of elements, and in fact they can be characterized up to isomorphism in second order logic. Even rather large portions of set theory can be described categorically by this logic.

Interesting as such a notion of logic is, it seems perfectly reasonable to distinguish the other sense of logic as a theory of deduction.[8] Elementary logic cannot characterize the usual mathematical structures, but rather seems to be distinguished by its completeness. Thus one is inevitably led to ask whether it is a necessary stopping point, or whether it can be extended to a richer logic which is still a theory of deduction in the same sense. Completeness, after all, is not just another nice property of a system. When a deductive system of whatever sort is presented, one of the most immediate questions is whether it is (in the relevant sense) complete. If all valid (or true) formulas can be proven by the rules, then apart from practical limitations such as length and complexity, they can be *known* to be valid (or true). This seems to be as interesting and significant a criterion as one could propose. Until the modern development of logic, it was generally assumed that mathematical systems had this property. One now knows from Gödel's work that the condition is too stringent even for number theory, and it is conceivable that it could have been too stringent for logic. Thus one should not claim to see *a priori* that a logic must be complete in order to be a theory of deduction. The point is rather that when it is discovered that the best known candidate satisfies such a condition, that tends to establish a sense of logic and a standard to be applied to competitors.

Two points concerning completeness should be mentioned. The mere existence of an effective enumeration of the valid formulas does not, by itself, provide knowledge. For example, one might be able to prove that there is an effective enumeration, without being able to specify one. Normally one will exhibit axioms known to be valid and rules known to preserve truth; preferably these axioms and rules will be more or less self-evident. Unfortunately these further conditions do not appear amenable to exact treatment. A second point, frequently noted, is that the completeness proof for *EL* actually shows that a somewhat vague intuitive notion, "valid *EL* formula", coincides with formal provability. It appears from inspection of the axioms and rules that formally provable formulas are intuitively valid; and intuitively valid formulas are certainly true in (say) all arithmetical models. Since the completeness theorem demonstrates that all formulas true in all arithmetical models are provable in *EL*, one must conclude that all three notions coincide in extension.

It is not out of the question that even in a theory of deduction completeness might be sacrificed for other advantages, such as greater expressive power. The strongest example historically is perhaps second order logic. One might claim that this is in some sense an acceptable theory of deduction, since in particular it yields all of the inferences of *EL*. But in fact it is not accepted as the basic logic. The expressive power of this logic, which is too great to admit a proof procedure, is adequate to express set-theoretical statements. Typical open questions, such as the continuum hypothesis or the existence of big cardinals, are easily stated as questions of the validity of second order formulas. Thus the principles of this logic are part of an active and somewhat esoteric area of mathematics. There seems to be a justifiable feeling that this theory should be considered mathematics, and that logic – one's theory of inference – is supposed to be more self-evident and less open.

Of course not all incomplete extensions of *EL* are as strong as second order logic. But the other known examples also tend not to look as natural, and they, like second order logic, invite further extension. In general, if completeness fails there is no algorithm to list the valid formulas; so one can expect many of the principles of the logic to be unknowable, or determinable only by means of *ad hoc* or inconclusive arguments. Clearly one will hesitate to substitute other desirable features for completeness in a theory of deduction. The negative evidence, together with the epistemological appeal of the completeness condition, make it seem reasonable to suppose that completeness is essential to an important sense of logic.

Strangely, compactness seems to be frequently ignored in discussions of the philosophy of logic. It is strange since the most important theories have infinitely many axioms. With only completeness it seems possible, *a priori*, that a logic might not prove all logical consequences of these theories. Compactness amounts to the condition that if X l.i. $\mathscr{A}$ then Γ l.i. $\mathscr{A}$ for some finite subset Γ of X. Since completeness ensures that if Γ l.i. $\mathscr{A}$ then $\Gamma \vdash \mathscr{A}$, one may conclude that if the system is both compact and complete, all logical consequences of a set of hypotheses are provable. We claim that that is the philosophical point at issue: if something follows, it can be known to follow.

However, the primary question is whether compactness by itself can be given a better justification than was given for completeness, since we are concerned to justify Lindström's hypotheses which require either completeness or compactness (plus the Löwenheim–Skolem condition). It seems to me that it is not all clear that compactness, *per se*, can be defended. The compactness condition in effect states that if $\mathscr{A}$ is implied by an infinite set of assumptions, $\mathscr{A}$ is already implied by a finite subset. The notion of implication here is of course the semantical one, not provability. The condition thus seems to state some weakness of the logic (as if it were futile to add infinitely many hypotheses), without yielding a compensating reward – such as knowledge that $\mathscr{A}$ is implied. To look at it another way, compactness also immediately entails that formalizations of (say) arithmetic will admit nonstandard models.

A second question, not strictly relevant to the present purpose, is whether, having accepted completeness, compactness can be seen to be an essential property of a theory of deduction. Perhaps one should first remark that to some extent completeness implies compactness,[9] so in many cases the question is dissolved, if not answered. We have noted that compactness is *sufficient* in conjunction with completeness to yield the desired consequence: if X l.i. $\mathscr{A}$ then $X \vdash \mathscr{A}$. It would appear that if one had a proof procedure of the usual sort the natural further condition to demand would be compactness, in order to get a reduction from an infinite set of hypotheses to a finite subset. The trouble is, it is not quite clear that one need divide the labor in exactly this way. For example, one might have a condition like: if X l.i. $\mathscr{A}$ then $\exists e(W_e \subseteq X$ and W_e l.i. $\mathscr{A})$, where W_e are the recursively enumerable sets (in a standard coding). If one also had an effective method for enumerating the pairs $\langle e, \mathscr{A}\rangle$ such that W_e l.i. $\mathscr{A}$, these two properties would do much the same work as completeness plus compactness. However, it is easy to show that they in fact imply compactness. This is no proof that compactness is necessary, but it seems highly likely that if completeness is required, compactness will be accepted. As for the main point, though, compactness by itself seems much less defensible than completeness.

This leaves the Löwenheim–Skolem property to be considered. On the face of it, this property seems to be undesirable, in that it states a limitation concerning the distinctions the logic is capable of making. Such a logic fails to make those distinctions intended in the usual theories, for example that there are uncountably many reals ("Skolem's paradox"). It is true that completeness also entails a limitation on the power of a logic to discern structures – a complete logic cannot, for example, determine the natural numbers by a single formula. But unlike completeness, the Löwenheim–Skolem condition does not express any clearly desirable property of a theory of deduction. It might follow from some other conditions which express desirable properties; in that case, it would be those conditions which one should attempt to defend. As it happens, it does follow from certain conditions which I shall attempt to justify (The existence of true but unprovable sentences in arithmetic is undesirable; it follows, however, from a defensible condition, namely that the axiom system be effective.)

There are positions, perhaps some kinds of finitism or countabilism, where the Löwenheim–Skolem property is, if not desirable, simply true. Obviously, under the general assumptions we have made, this neatly resolves the question of what logic is. However these positions are not merely minority positions, but positions which ignore, or drastically reinterpret, the overall body of science and mathematics. Further, I doubt whether the arguments that there are only countably many things are very cogent in their own right. I cannot pursue a discussion of these related positions here, but will simply accept mathematics as it exists, and conclude that there is no *a priori* reason to impose the Löwenheim–Skolem condition on a logic.

Some Complete Competitors

Two examples of axiomatizable logics have been discussed in the literature. The first, call it $L(U)$, adds an additional quantifier (Ux), which is read "for uncountably many x". Some time after the logic was known to be axiomatizable, Keisler[10] proved that the following elegant set of axioms is adequate:

0 Axioms of *EL*.
1 $(\forall x)(\forall y)\neg(Uz)(z = x \vee z = y)$, "The uncountable is bigger than 2".
2 $(\forall x)(\mathscr{A} \rightarrow \mathscr{B}) \rightarrow [(Ux)\mathscr{A} \rightarrow (Ux)\mathscr{B}]$, "If all $\mathscr{A}$s are $\mathscr{B}$s, and there are uncountably many $\mathscr{A}$s, then there are uncountably many $\mathscr{B}$s."
3 $(Ux)\mathscr{A}(x) \leftrightarrow (Uy)\mathscr{A}(y)$, "Changing variables".
4 $(Ux)(\exists y)\mathscr{A} \rightarrow [(\exists y)(Ux)\mathscr{A} \vee (Uy)(\exists x)\mathscr{A}]$, "If uncountably many things are put into boxes, either some box gets uncountably many members, or else uncountably many boxes are used".

Modus ponens and generalization are the rules of inference. Notice that all of the axioms remain valid if the quantifier "for infinitely many x" is substituted for (Ux). It is easy, however, to show that there are formulas valid under this interpretation which are not valid under the original interpretation. Also note that the axioms do not assert that there are uncountably many things.

By considering the contrapositive of Axiom (4), one sees that it expresses the principle that a countable union of countable sets is countable. This may be considered a weak form of the axiom of choice. Dropping choice, one can give models for Zermelo–Fraenkel set theory in which a countable union of countable sets is uncountable. On the other hand, if one assumes certain forms of the axiom of choice, such as the principle that a countable set of nonempty sets has a choice function, then (4) follows readily.

Since a version of the axiom of choice is expressed by the validity of a formula of $L(U)$, if one accepts $L(U)$ one accepts this form of the axiom of choice as a principle of logic. This is especially interesting in view of the rather controversial history of this axiom. The objections against it, however, now seem simply mistaken. Probably the main objection was that it was supposed that the choice function had to be given by some law or definition. But since sets are completely arbitrary collections, this requirement is irrelevant. Except for formalists, who regard it as meaningless, set theorists generally regard the axiom of choice as true, and indeed practically obvious. Whether or not it may be taken as a logical principle is another matter – its infinitistic nature might give pause.

The second example, L (C), has the quantifier (Cx), the Chang quantifier:[11] (Cx) $\mathscr{A}(x)$ is satisfied in a model just in case $\{x : \mathscr{A}(x) \text{is satisfied}\}$ has the same cardinality as the universe of the model. This logic, however, is complete and compact only if one rules out finite models. One does not have a simple and explicit axiom system for the Chang quantifier, but there is one for this logic with identity deleted.[12] Curiously, it too assumes a form of the axiom of choice. It would be interesting to know whether there is some deeper connection between axiomatizable extensions of *EL* and the axiom of choice. It is possible to give rather trivial extensions of *EL* and prove them complete and compact without using the axiom of choice. Whether this is true for any interesting extensions seems to be an open question.

Recently other examples of axiomatizable logics have been found which serve also to illustrate another generalization of quantifiers. Besides quantifiers of one argument, one might want quantifiers of two arguments. For example (Wx,y) $\mathscr{A}(x,y)$ might mean that $\mathscr{A}(x,y)$ defines a well ordering. This is a useful concept, but like (Ix), it will enable one to characterize arithmetic categorically and so cannot give a complete logic. Shelah[13] has shown that there is a whole category of quantifiers similar to (Wx, y) which give axiomatizable logics. Unfortunately they involve technical notions of set theory: for any regular cardinal λ let $(S^\lambda x,y)$ $\mathscr{A}(x,y)$ mean that $\mathscr{A}(x,y)$ defines a linear ordering of cofinality λ. The quantifiers S^λ give axiomatizable logics, and moreover one can generalize to certain sets of cardinals, and to the quantifiers "confinality less than λ", and still get axiomatizable logics.

One result to hope for is that there might be a maximum logic L, that is, an axiomatizable logic containing all others. This is immediately seen to be impossible since any logic L has only countably many L-classes, while the various logics of Shelah define an unlimited number (i.e. a proper class in number) of classes of models. More interesting is the observation that our first two examples are incompatible: L (U) and L (C) have no common extension.[14]

One might wonder how these extended logics could or would be used. Consider number theory. An obvious axiom to state using (Ux) is that there are only countably many numbers. Or, using the Chang quantifier, one can say that the set of predecessors of a number is of smaller cardinality than the universe. It seems possible that these axioms might yield new theorems in the original language of number theory, but they do not. The proof is easy for the first axiom, but is not at all trivial for the second.[15] In number theory there is no reason to suppose that all such proposals will yield conservative extensions. However, in set theory it is clear that one will never get new theorems, at least if the axioms of the logic are set theoretical principles. This is because any proof in the strong logic can be translated into an ordinary proof which uses principles of set theory. Although it is conceivable that some logical axiom independent of set theory could be seen to be true, it is probable that it would be clearly a set theoretical principle. Thus extended logics are probably best regarded as changes in the boundary which demarcates as logic a part of set theory.

Can the Competitors Be Rejected?

There is a serious objection against the quantifier "for uncountably many x". One would expect that if this were a legitimate quantifier, the quantifier "for infinitely many x" would also be acceptable; but no complete logic can contain the latter quantifier. Specifically, the cardinal $\aleph_1$ plays a role in the model theory of any complete logic containing the quantifier (Ux) which cannot be played by the smaller cardinal $\aleph_0$. One can say "The universe has cardinality less than $\aleph_1$", but one cannot say "The universe has cardinality less than $\aleph_0$". One cannot even have a formula, using various predicate letters, which is satisfiable exactly in finite universes. The fact that one can say so much more about $\aleph_1$ than $\aleph_0$ seems to me to be a state of affairs sufficiently unnatural to discredit (Ux) as a logical notion.

If the uncountable is no logical notion, a line of argument is suggested which invokes Lindström's theorem. This theorem entails that any complete extension of elementary logic must have something like an "axiom of uncountability", that is, it must have a formula with uncountable models but no countable models. If the uncountable is no logical concept, one is tempted to regard this consequence as a proof that there are no complete logics extending elementary logic. However that would be a mistake. Elementary logic already has "axioms of infinity" in a similar sense. That is, there are formulas with models of all infinite cardinalities but with no finite models, even though "infinite" is not a logical concept in the sense of there being a quantifier expressing "for infinitely many x".

Being able to distinguish the uncountable by means of axioms of uncountability is evidently a much weaker property than having a quantifier "for uncountably many x". Further, if one allows even weaker criteria, elementary logic is already able to distinguish the uncountable from the countable. Call a theory with infinite models categorical in the cardinal $\aleph$ if all models of cardinality $\aleph$ are isomorphic. It is known that there also theories categorical in $\aleph_0$ but in no uncountable cardinal. There are also theories categorical in all uncountable cardinals but not in $\aleph_0$; theories categorical in all infinite cardinals; and theories categorical in no infinite cardinals. Morley's theorem demonstrates that for infinite cardinals, these are the only possible categories. In this sense, elementary logic cannot distinguish two uncountable cardinals, although it can distinguish the countably infinite from the uncountably infinite.

From this extensional viewpoint, Shelah's quantifier "cofinality ω" fares considerably better than the quantifier (Ux). One has axioms of uncountability in precisely the same sense one has axioms of infinity, namely formulas with predicate letters satisfiable in, and only in, the uncountable universes. Other gross properties seem reasonable also: just as one cannot have formulas satisfiable exactly in finite universes, this logic has no formulas satisfiable exactly in countable universes. The obvious objection against the Shelah logic is that one would never have supposed that a technical notion like cofinality was a logical concept. Unless it could be shown equivalent to some more palatable notion, this must remain a serious objection.

Continuity of the Standard Quantifiers

If one considers, instead of the entire logic EL, the standard quantifiers, it would appear that there must be some sense in which $\forall$ and $\exists$ are very simple and primitive. It may be possible to state some natural condition which expresses this simplicity, and which, at the same time, rules out quantifiers one feels are no part of logic. To start with, $\forall$ and $\exists$ may be regarded as extrapolations of the truth functional connectives $\wedge$ and $\vee$ to infinite domains. To see how one can make such extrapolations, take the infinite list of sentential letters $P_0, P_1, P_2, \ldots$, and suppose one is given an arbitrary truth assignment t, that is, a function which assigns a truth value $t(P_i)$ to each letter P_i. By the truth table rules, t may be extended to assign a value $t(\mathscr{A})$ to each formula $\mathscr{A}$ of sentential logic.

Consider, for increasing n, the values $t(P_0 \wedge P_1 \wedge \ldots \wedge P_n)$ assigned to the finite conjunctions. One sees at once that no matter what assignment t is, the limit $\lim n \to \infty\, t(P_0 \wedge P_1 \wedge \ldots \wedge P_n)$ has a clear value: there is a finite point N such that the conjunction $(P_0 \wedge P_1 \wedge \ldots \wedge P_N)$ is assigned a value $t(P_0 \wedge P_1 \wedge \ldots \wedge P_N)$, and for all m greater than $N, t(P_0 \wedge P_1 \wedge \ldots \wedge P_m)$ is the same as $t(P_0 \wedge P_1 \wedge \ldots \wedge P_N)$. It is easily seen that exactly the same property is true of finite disjunctions.

Compare another familiar binary connective, the biconditional $\leftrightarrow$. It is also commutative and associative so that parentheses may be dropped, and $(P_0 \leftrightarrow P_1 \leftrightarrow \ldots \leftrightarrow P_n)$ considered to be a formula. However, in this case there is no evident limit of the values $t(P_0 \leftrightarrow P_1 \leftrightarrow \ldots \leftrightarrow P_n)$ as n increases. This is not merely due to the unfamiliarity of the construction. If, for example, $t(P_i)$ is $\perp$ (falsehood) for all i, then the value of $t(P_0 \leftrightarrow P_1 \leftrightarrow \ldots \leftrightarrow P_n)$ is $\top$ for n even, and $\perp$ for n odd. There is simply no well-defined limit value to be used for an infinite quantification.

These considerations can be restated in the language of quantifiers and predicates. Suppose a model M is given which interprets a one-place letter F. In each finite submodel $\mathcal{J}$ with universe $\{a_0, \ldots, a_n\}, \forall x F(x)$ has a truth value, the same value as $(F(a_0) \wedge F(a_1) \wedge \ldots \wedge F(a_n))$ has in $\mathcal{J}$. $\forall$ is continuous in the sense that for each model M there is a finite submodel $\mathcal{J}$, in which $\forall x F(x)$ takes a truth value and holds that truth value for all submodels K between $\mathcal{J}$ and M, including M. Thus if one thinks of the model M as being revealed step by step, there is a finite portion at which $\forall x F(x)$ assumes a truth value, and holds that truth value no matter how much the model is further revealed, and even if it is totally revealed.

This is the continuity condition we wish to isolate, and of course the quantifier $\exists$, as well as

$\forall$, is continuous in this sense. However if (Qx) is any quantifier such that $(Qx)F(x)$ agrees with $(F(a_0) \leftrightarrow F(a_1) \leftrightarrow \ldots \leftrightarrow\leftrightarrow F(a_n))$ in models with universe $\{a_0, a_1, \ldots, a_n\}$, then (Qx) must exhibit a sort of discontinuity: For certain M, (Qx) $F(x)$ will oscillate back and forth in truth value as one considers larger and larger finite submodels, and the value of (Qx) $F(x)$ in infinite models will be no direct extrapolation of its value in finite submodels. It should be noted in passing that there is another possible kind of discontinuity which occurs, for example, with the quantifier "for infinitely many x". Since (Ix) $(x = x)$ comes out false in all finite models, it has a definite limiting value – the trouble is that its value in an infinite model is not equal to this limit.

We have exhibited a type of continuity for the basic quantifiers $\forall$ and $\exists$ in terms of the behavior of the formulas $\forall xF(x)$ and $\exists xF(x)$, where F is any one-place letter. Not only are the basic quantifiers continuous, but every formula of elementary monadic logic, which is the subsystem of *EL* using one-place relations, is continuous in this sense. Moreover one has an exact converse: Given any quantifier[16] of monadic type, if it is continuous in the sense sketched above, it is definable in elementary monadic logic.

It is clear that the standard quantifiers can be *tested* for truth in certain ways. Let us examine more closely the exact sense in which $\exists$ has operational meaning. Suppose one has an intuitively decidable one-place predicate F, interpreted (say) as "is a frog". One wants to determine the truth of $\exists xF(x)$ in the standard model – the actual world. One proceeds along, examining specimens, and at some finite point one reaches a portion of the model in which $\exists xF(x)$ comes out true; and at that point one knows that it is true in the whole model no matter what the rest of the model is like. (We are assuming, for purposes of the example, that the world is infinite.) It is easy to formulate this criterion precisely and show that any such quantifier is continuous.[17]

This test does not work for $\forall$; instead, if $\forall xF(x)$ is ultimately false, its falsehood can be known at a finite point. Another way to put it is that one has only half a test for $\exists$: if $\exists xF(x)$ is true, one can actually find it out in finitely many steps; but if $\exists xF(x)$ is false, one may never be sure, short of exhausting the entire model. One might consider the stronger demand that the correct truth value always be ascertainable at a finite point, even though $\forall$ and $\exists$ do not satisfy such a stringent condition. But it is easy to show that only trivial quantifiers result.

In summary, those quantifiers which are partially decidable in the same sense as $\exists$ are all continuous; their complements are the quantifiers partially decidable in the same sense as $\forall$, and are also continuous. The notion of continuous quantifier seems to be the most natural symmetrical condition containing both $\forall$ and $\exists$. We have remarked that any continuous *monadic* quantifier is definable in elementary monadic logic, hence in *EL*. For non-monadic continuous quantifiers, however, one must appeal to completeness of the resulting logic to conclude that they are definable in *EL*. In fact, one can give binary quantifiers which are partially decidable in the same sense as $\exists$, but which give incomplete logics. It would seem, in conclusion, that the major properties of the standard quantifiers of epistemic significance can be precisely captured, and any such quantifier shown to be definable in *EL*, invoking completeness only for non-monadic quantifiers.[18]

What about Complex Formulas?

One sees at once that no such simple criteria as continuity apply to an arbitrary formula of the full elementary logic of relations. For example, there is a formula $\mathscr{A}$, with a single binary letter G, which says G is a linear ordering without last element. $\mathscr{A}$ has only infinite models, so given a model M of $\mathscr{A}$, $\mathscr{A}$ must be false in each finite submodel of M. One can also exhibit other kinds of discontinuity: There is a formula $\mathscr{B}$ which says that G is a function mapping part of the universe one-one onto the remainder; $\mathscr{B}$ has infinite models, and for any such model N, $\mathscr{B}$ will be true in some finite submodels, and false in others.

Continuity does not apply to arbitrary complex formulas of *EL*, and the best one can say is that a complete logic based on continuous quanti fiers does not extend *EL*. Whether this formulation is good enough is open to question. How does one, for example, know that a given logic can be defined by adjoining quantifiers to *EL*? Put this way, the question does not pose a problem. We have taken a quantifier to be an arbitrary class of models of a fixed type, closed under isomorphism of models, which amounts to saying that a totally arbitrary formula can be taken to be a quantifier (see note 18). The assumption that the logic be based on quantifiers seems to involve no great restriction, for, given a logic L, one can take each formula of L as a quantifier and adjoin it to EL, getting L^*. Trivially, $L \subseteq L*$, and it is reasonable to suppose $L = L*$, because otherwise L would not be closed under the familiar constructions with connectives and the iteration of quantifiers.

The problem, then, is not that it is unreasonable to assume a given logic L is constructed by adjoining quantifiers to EL. Rather, the problem is to give some intrinsic justification for the particular method by which complex formulas are built up from the primitives. That is, suppose one is presented with a logic L, defined by adjoining quantifiers to EL, but which cannot be defined by the adjunction of continuous quantifiers to EL. How does one know that there might not be some other acceptable way of generating the logic from simple quantifiers? Second order logic is in a sense generated from the standard quantifiers – but one applies them to predicate letters as well as to individual variables.

Complex formulas with relations no longer have the operational simplicity of the basic quantifiers, or the finitistic nature of monadic formulas. Thus it is not easy to see a suitable criterion to apply directly to arbitrary formulas. And, in the absence of such a criterion, it is not easy to predict what primitives and operations might have a clear enough meaning to be used in constructing a logic.

These seem to me to be serious objections which pinpoint the weakness in this argument for the primacy of EL. I do not know how to overcome them, nor am I confident that there is a completely watertight argument for EL. Everything considered, I think it must be conceded that the considerations which gave a rather satisfactory characterization of elementary monadic logic do not provide a comparably definitive characterization of the full logic. But they do, I believe, give some insight into the reasons EL has been taken as standard, and the reasons it appears natural and primitive in comparison with the known extensions.

Logic and Ontology

The reasons for taking elementary logic as standard evidently have to do also with certain imprecise – but I think *vital* – criteria, such as the fact that it easily codifies many inferences of ordinary language and of informal mathematics, and the fact that stronger quantifiers can be fruitfully analyzed in set theory, a theory of EL. It is not surprising that in practice elementary logic has been taken as logic. Yet the criteria which justify this choice do not justify the use to which elementary logic is frequently put. It should be recalled that a tremendous amount of weight has been thrown on the alleged distinction between logic (i.e. elementary logic) and mathematics. Perhaps the most extreme example is Skolem,[19] who deduces from the Löwenheim–Skolem theorem that "the absolutist conceptions of Cantor's theory" are "illusory". I think it clear that this conclusion would not follow even if elementary logic were in some sense the true logic, as Skolem tacitly assumed. From the absolutist standpoint, elementary logic is not able to preserve in the new structure all significant features of the initial structure. For example, the power set of ω is intended to contain *all* sets of integers. It should also be noted that one has a similar problem for ordinary number theory, and even for certain weak decidable subtheories of number theory: there are nonstandard models, countable as well as uncountable.

Elementary logic has also been invoked in connection with ontology by Quine, who has argued that one is to look to the range of the quantifiers to uncover one's ontological commitments. There are difficulties in interpreting this prescription which we shall not dwell on here, but at a very basic level one can question his doctrine by proposing different logics. One challenge he considers[20] is a logic due to Henkin, which has formulas with "branching quantifiers" such as $\begin{smallmatrix}\forall u \exists v\\ \forall x \exists y\end{smallmatrix} F(u,v,x,y)$, which is to be interpreted, using Skolem functions, as $\exists f \exists g \forall u \forall x F(u, fu, x, gx)$. This looks very much like the Skolem form of a formula of EL, the difference being that v is a function of u alone, and y is a function of x alone. To express such a formula in EL one naturally quantifies over functions. Thus such a formula as $\begin{smallmatrix}\forall u \exists v\\ \forall x \exists y\end{smallmatrix} F(u,v,x,y)$ appears to blur the distinction between those objects which are quantified over, and those which are not. Quine rejects the Henkin logic primarily[21] because it is not complete (it turns out, surprisingly, that in it one can express the quantifier "for infinitely many x"). But we have a number of "deviant" logics which are axiomatizable and allow us to do exactly this sort of thing: we can say that there are countably many helium atoms without quantifying over anything except physical objects. Quine does concede that some extensions of EL are complete, and mentions the example of a logic which adds finitely many valid formulas. However, no indication is given how the complete extensions of *EL* are to be ruled out.

To go to another extreme, one can formulate *EL* with modus ponens as the sole rule of inference. By means of a simple translation one may consider *EL*, and theories based on it, to be theories in *sentential* logic. Sentential logic certainly has many attractive technical properties to recommend it. And perhaps one's ontological commitment would be to the two truth values.

This is clearly an implausible suggestion, and it is not hard to see why. Evidently our conceptual scheme is such that we think of the world in terms of objects and relations. Sentential logic deals with whole sentences and, unlike *EL*, suppresses this prior analysis and *prior commitment*. Of course *EL* has quantifiers as well as individual variables and relation letters. The particular choice of quantifiers must be explained, and we have attempted to give illuminating reasons why the standard quantifiers are singularly primitive. One can consider stronger quantifiers, but one does not have as clear a grasp of their meaning, and they usually seem to demand further explanation.

This is not to claim that it is impossible to operate with certain quantifiers, such as "for infinitely many x", or equivalently, "for finitely many x". It is true that in this case the resulting logic is not complete and invites further extension. These are good reasons to conclude that the logic is not as satisfactory as *EL*, but they do not seem to bear on the ontological issue. Perhaps one could understand "for finitely many x" in some intuitive sense as a primitive notion and work quite well within this logic. Many of the rules would be sufficiently clear – for example if $\mathscr{A}(x)$ and $\mathscr{B}(x)$ are each true of finitely many x, so are $(\mathscr{A}(x) \wedge \mathscr{B}(x))$ and $(\mathscr{A}(x) \vee \mathscr{B}(x))$. The user of this logic might not have in mind any particular analysis of "finite". If so, it would seem incorrect to attribute to him some such analysis in terms of an ontology of sets, or of numbers and functions. Which of several possible analyses should be attributed to him?

It may even be fair to say that mathematicians in effect used such quantifiers before the development of set theory. Since Cantor and Dedekind, one has a reasonably clear and quite general theory of "finite", "infinite", and related notions. In view of the existence of such a general theory, there is little point in taking notions such as "finite" as primitive. The conceptual scheme of set theory deals with objects (sets), relations (membership and identity), and uses the standard quantifiers. Formulas of *EL* directly codify this scheme, and, so interpreted, they reflect ontological assumptions correctly. But to appeal to criteria such as completeness to justify the logic, and then mechanically use the logic to "assess a theory's ontological demands", seems to stand the matter on its head. It also leads, via the Löwenheim–Skolem theorem, to pointless puzzles about ontological reduction. These, however, have been discussed at length elsewhere.[22]

Notes

I am heavily indebted to Jonathan Lear, D. A. Martin and Hao Wang with whom I discussed many of these matters. I also wish to express thanks to a number of other colleagues and students who contributed valuable criticism and comments.

1 An *L*-class has models of a fixed finite type, e.g., with one binary and four ternary relations.

2 See A. Tarski, *Logic, Semantics, Metamathematics*, Clarendon Press, Oxford, 1956, pp. 409–20.

3 For monadic logics it does matter whether one assumes the Löwenheim–Skolem property for single formulas or for infinite sets of formulas. See my paper, "The Characterization of Monadic Logic", *The Journal of Symbolic Logic* 38 (1973), 481–8.

4 See Per Lindström "On Extensions of Elementary Logic", *Theoria* 35 (1969), 1–11. It should be noted that Harvey Friedman later rediscovered these theorems and pointed out their philosophical interest.

5 See "Arithmetical Extensions of Relational Systems", *Compositio Mathematica* 13 (1957), 81–102.

6 Something like quantification is used with regard to the "upward Löwenheim–Skolem property" (Theorem 3) and in Corollary 2 of Lindström's paper cited above.

7 D. A. Martin has emphasized this distinction to me. Of course there are other senses of logic, for example, as "a science prior to all others, which contains the ideas and principles underlying all sciences". See Kurt Gödel, "Russell's Mathematical Logic", in *Philosophy of Mathematics*, Benacerraf and Putnam (eds.), Prentice-Hall, Englewood Cliffs, N.J., 1964, p. 211.

8 I concentrate only on certain gross features of a theory of deduction. This is not to deny that finer structure – the particular rules of inference and axioms – may be highly important.

9 Lindström has shown in unpublished work that for a wide class of logics, if the logic is complete it is compact for recursively enumerable sets of formulas. The result holds for logics using finitely many "generalized quantifiers". Lindström's "First Order Predicate Logic with Generalized Quantifiers" (*Theoria* 32 (1966), 186–95) discusses such quantifiers; see also note 17 below.

10 See H. J. Keisler, "Logic with the Quantifier 'There Exist Uncountably Many'", *Annals of Mathematical Logic* 1 (1970), 1–93. This logic is *countably* compact; this is the sense in which we are using "compact", since sets of formulas are taken to be countable.

11 Named after C. C. Chang. It is sometimes called the "equicardinal quantifier", but this suggests the quantifier (Qx) which operates on a pair of formulas, $(Qx)\,[\mathscr{A}(x); \mathscr{B}(x)]$ being true in M just in case $\mathscr{A}(x)$ and $\mathscr{B}(x)$ are true of the same number of things in M. Although similar to (Cx), this quantifier is too strong to be axiomatizable. This follows because one can say, "G is a linear ordering with no last element, and any two distinct points have a distinct number of predecessors". This formula characterizes the ordering of the natural numbers.

12 See Bell and Slomson, *Models and Ultraproducts*, North-Holland, Amsterdam, 1969, p. 283.

13 Personal communication. These results are related to his "On Models with Power-Like Orderings", *The Journal of Symbolic Logic* 37 (1972), 247–67.

14 To prove $L(U)$ and $L(C)$ have no common extension, consider the formula which says "G is a linear ordering with no last element and $(\forall x)\neg(Cy)\,G(y,x)$". The only countable model of this formula is isomorphic to $\langle \omega, \langle \rangle$. Since one can say "the universe is countable" in $L(U)$, any logic containing $L(C)$ and $L(U)$ characterizes $\langle \omega, \langle \rangle$. Martin pointed out that one can modify the quantifier C to a quantifier C' which is C in universes of cardinality $\leq \aleph_1$, and is $\forall$ in larger universes. The generalized continuum hypothesis was used to prove only special cases of the axiomatizability of C, and it turns out it is not needed for C'. Also C' is axiomatizable without restriction to infinite models. Our argument above still holds for $L(U)$ and $L\,(C')$.

15 See Bell and Slomson, *op. cit.*, pp. 284–5.

16 A *type* is a finite sequence of predicate letters; a *monadic* type has only monadic letters. A *quantifier* $\mathscr{Q}$ is a class of models of some fixed type, such that if $M \in \mathscr{Q}$ and N is isomorphic to M, then $N \in \mathscr{Q}$. For example, suppose $\mathscr{Q}$ is the class of models of type $\langle G \rangle$, where G is binary and is interpreted in the model as a well-ordering of the universe. One adjoins this $\mathscr{Q}$ to EL by introducing a symbol Q, and adding to the definition of formula the clause: $Q_i, v_j \mathscr{A}(v_i, v_j)$ is a formula if $\mathscr{A}(v_i, v_j)$ is. $M \models Q v_i, v_j \mathscr{A}(v_i, v_j)$ is defined to mean $M* \in \mathscr{Q}$ where the universe of $M*$ is the universe of M, and the binary relation of $M*$ is $\{\langle x_i, x_j \rangle : M \models \mathscr{A}(x_i, x_j)\}$. Thus $M \models Q v_i, v_j\ \mathscr{A}(v_i, v_j)$ just in case $\mathscr{A}(v_i, v_j)$ defines a well ordering over the universe of M; this Q is just the quantifier W mentioned in passing in Section 3. For further discussion see my "Continuity and Elementary Logic", *The Journal of Symbolic Logic* 39 (1974), 700–16.

17 Let $\mathscr{P}$ be a class of finite models satisfying: if $M \in \mathscr{P}$ and N is isomorphic to M then $N \in \mathscr{P}$. A continuous quantifier $\mathscr{Q}$ is defined by: $M \in \mathscr{Q} \Leftrightarrow (\exists J)[J \subseteq M \wedge J \in \mathscr{P}]$. To bring in considerations of effectiveness, encode each finite model J with universe $\{0, 1, \ldots, n\}$ as a number $\#\,(J)$. Given a recursive set R, define $\mathscr{P} = \{K : (\exists J)\,[K$ is isomorphic to J and $\#\,(J) \in R]\}$, and define $\mathscr{Q}$ as above. Then given a model M, suppose that one examines finite submodels K, checking whether there is an isomorphic J with $\#(J)$ in R. M is in $\mathscr{Q}$ if and only if this procedure eventually leads to a positive answer. Conversely, any quantifier which is partially decidable in the same sense as $\exists$ appears to fall under this definition, for suitably chosen recursive R.

18 A logic based on continuous quantifiers satisfies the Löwenheim–Skolem condition, so if it is complete, it is EL. Note that if one demands a uniform finite bound (uniform continuity) one need not invoke completeness. See especially theorems 2, 4, 5 and 7 of my paper cited in note 17.

19 See p. 47 of Thoralf Skolem, *Abstract Set Theory* (Notre Dame Mathematical Lectures No. 8, Notre Dame, 1962).

20 See pp. 89–91 of W. V. Quine, *Philosophy of Logic*, Prentice-Hall, Englewood Cliffs, N.J., 1970.

21 However, in an earlier paper Quine also notes that a certain typical construction which one would formulate with the Henkin quantifier "is not after all very ordinary language; its grammar is doubtful". See p. 112 of "Existence and Quantification", in *Ontological Relativity and Other Essays*, Columbia University Press, New York, 1969.

22 See my "Ontological Reduction", *The Journal of Philosophy* 68 (1971), 151–64.