THREE-VALUED LOGIC AND FUTURE CONTINGENTS

What might now be called the ' classical ' system of three-valued logic was introduced by Lukasiewicz in 1920, further discussed (along with other many-valued systems) by Lukasiewicz and Tarski in 1930, and axiomatised by Wajsberg in 1932. Outlines of this system have appeared in English, notably in Lewis and Langford's *Symbolic Logic*[1] and in Dr. Jordan's monograph on *The Development of Mathematical Logic and of Logical Positivism in Poland between the Two Wars* ;[2] but in Lewis and Langford, our fullest and most accessible English source, Lukasiewicz's own very neat notation is unfortunately altered, so that we do not see the formal features of the system in their full clarity, while Dr. Jordan's discussion is relatively sketchy. In neither place, moreover, is the system clearly related to the problem by which it was first suggested—the problem of the truth-value of propositions about contingent future events, as raised in Aristotle's *De Interpretatione*. (Jordan simply mentions Lukasiewicz's preoccupation with this problem, while Lewis gives a detailed interpretation of the system, but one which has hardly anything to do with ' future contingents '). There is room, therefore, for a little more to be said on the subject.

Most of us—this is certainly true of myself—have a strong initial repugnance to the whole conception of a three-valued logic, a repugnance not unlike that which an earlier generation seems to have felt towards systems of material implication. And the repugnance probably springs in both cases from the same source—a failure (partly fostered by over-pugnacious advocates of the system) to understand just what is being talked about, and a tendency to confuse what is being talked about with something else for which the theses being put forward are plainly untrue. What needs to be shown, in order to remove this repugnance, is that, in so far as the system is designed to be interpreted at all (and I do not think it necessary or desirable to eschew interpretation altogether), it is designed to handle what are not unlike ' propositions ', ' truth-values ', relations of material implication, etc., which we meet with in two-valued systems, but what are nevertheless not quite the same. I shall begin, however, by considering the system in its purely formal aspect.[3]

The truth-values of which the propositions of the system are considered to be capable are truth, symbolised by ' 1 ', falsehood, symbolised by ' 0 ',

[1]Ch. VII (This chapter is the work of Professor Lewis).

[2]*Polish Science and Learning*, No. 6 (Oxford, 1945), pp. 27 ff.

[3]In the paragraphs which follow I shall briefly explain the symbolic techniques employed as I go along, as the Polish notation is only beginning to be familiar in English-speaking countries. It is the same general type of notation as that explained in Lukasiewicz's *Aristotle's Syllogistic* (Oxford, 1952), §§22 and 23. (See this also for the setting-out of truth-table calculations and of proofs).

and a third, symbolised by '$\frac{1}{2}$'. From propositions thus considered we may form various truth-functions, i.e. new propositions the truth-values of which depend solely on the truth-values of their components. In the axiomatised system the undefined functions are 'Np' (roughly 'Not-p') and 'Cpq' (roughly 'If p then q'), which have the properties indicated by the following equations :—

(i) $N1 = 0$; $N\frac{1}{2} = \frac{1}{2}$; $N0 = 1$;

(ii) $C11 = C\frac{1}{2}\frac{1}{2} = C00 = C\frac{1}{2}1 = C01 = C0\frac{1}{2} = 1$;

$C1\frac{1}{2} = C\frac{1}{2}0 = \frac{1}{2}$;

$C10 = 0$;

or by the following truth-tables :—

N	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

C	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

(Either device tells us that when $p = 1$, i.e. is true, $Np = 0$, i.e. is false ; when $p = \frac{1}{2}$, $Np = \frac{1}{2}$; when $p = 0$, $Np = 1$; when p and q are both 1, $Cpq = 1$; when p is 1 and q is $\frac{1}{2}$, $Cpq = \frac{1}{2}$; and so on). In terms of 'C' and 'N' we may define the further functions 'Apq' (roughly 'Either p or q'), 'Kpq' (roughly 'Both p and q') and 'Epq' (roughly 'If and only if p then q') as follows :—Apq = CCpqq ; Kpq = NANpNq ; Epq = KCpqCqp. The definitions of 'Both p and q' as 'Not either not-p or not-q' and of 'If and only if p then q' as 'Both if p then q and if q then p' are met with in two-valued systems also, e.g. in *Principia Mathematica.* The definition of Apq is peculiar, and will be commented on in a moment. These definitions give the properties indicated by the following tables :—

A	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

K	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

E	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

We may obtain these from the definitions together with the tables or equations for C and N by calculations of this sort :—

$$A11 = CC111 = C11 = 1$$
$$A1\tfrac{1}{2} = CC1\tfrac{1}{2}\tfrac{1}{2} = C\tfrac{1}{2}\tfrac{1}{2} = 1$$
$$A10 = CC100 = C00 = 1.$$

(Taking the second line, we have $A1\frac{1}{2} = CC1\frac{1}{2}\frac{1}{2}$ because Apq is defined as CCpqq ; and $CC1\frac{1}{2}\frac{1}{2}$ gives $C\frac{1}{2}\frac{1}{2}$ because we have $C1\frac{1}{2} = \frac{1}{2}$ in our equations for C ; and those equations also give us $C\frac{1}{2}\frac{1}{2} = 1$).

Wajsberg has shown that any formula which works out by calculation from the truth-tables as a logical law (i.e. as true for all possible values of the p's, q's etc. contained in it) can be formally deduced, by the ordinary rules of substitution and detachment, from the following four axioms : 1. CpCqp (If p then if q then p), 2. CCpqCCqrCpr (If p implies q, then if q implies r, p implies r), 3. CCCpNppp (If if-p-then-not-p implies p, then p), and 4.

CCNpNqCqp (If not-p implies not-q, q implies p).[4] All these work out as laws by the truth-tables ; for example, with 3 we have

$$CCC1N111 = CCC1011 = CC011 = C11 = 1$$
$$CCC\tfrac{1}{2}N\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} = CCC\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2} = CC1\tfrac{1}{2}\tfrac{1}{2} = C\tfrac{1}{2}\tfrac{1}{2} = 1$$
$$CCC0N000 = CCC0100 = CC100 = C00 = 1.$$

And the table for C is such that the ordinary rule of detachment may be safely applied ; that is, in this sense of ' implication ' as in others, what is implied by a true proposition is true (for the only case in which, when p = 1, Cpq also = 1, is that in which q = 1). As an illustration of a deduction from these axioms we might offer the following :—

1. CpCqp
2. CCpqCCqrCpr
3. CCCpNppp
   1 q/CpNp = 5.
5. CpCCpNpp
   2 q/CCpNpp, r/p = C5 — C3 — 6
6. Cpp.

(The line ' 1 q/CpNp = 5 ' tells us that the substitution of CpNp for q in axiom 1 will yield the new law 5 ; while the other derivational line tells us that the substitution of CCpNpp for q and of p for r in axiom 2 will yield a long double implication in which the first antecedent is the law 5, the second antecedent the axiom 3, and the consequent the new law 6, C3 — 6 being detachable as a law because the antecedent 5 is a law, and 6 because 3 is a law. 6 is of course the law of identity, ' If p then p ', which is a law in this system as in others[5]).

Returning to the truth-tables, it will be found that if we draw a diamond around the values for cases in which one or both of the arguments has the value $\tfrac{1}{2}$—for example round the group

$$\begin{matrix} & 1 & \\ 1 & \tfrac{1}{2} & \tfrac{1}{2} \\ & \tfrac{1}{2} & \end{matrix}$$

in the table for A—the remaining values will be those which appear in the tables for the corresponding functions in ordinary two-valued logic, where ' If ', ' Either ', ' Both ' and ' If and only if ' have the properties indicated by the tables

C	1	0
1	1	0
0	1	1

A	1	0
1	1	1
0	1	0

K	1	0
1	1	0
0	0	0

E	1	0
1	1	0
0	0	1

Comparing the two sets of tables, we may say that in three-valued logic as in two-valued, Apq is true if and only if one of the alternatives is true, and false if and only if both alternatives are false ; Kpq is true if and only if both of its parts are true, and false if and only if one of them if false ; Cpq

[4] See J. B. Rosser and A. R. Turquette, ' Axiom Schemes for m-valued Propositional Calculi ', *Journal of Symbolic Logic*, Sept. 1945.

[5] The proof above may be compared with Lukasiewicz's proof of Cpp in a two-valued system in *Aristotle's Syllogistic*, p. 81.

is true if and only if its consequent has at least as great a truth-value as its antecedent, and false if and only if its antecedent is true but its consequent false; and Epq is true if and only if its parts have the same truth-value, and false if and only if one part is true and the other false. Three-valued logic differs from two-valued, however, in that the application of these criteria does not make Cpq, the material implication of q by p, equivalent to ANpq, but makes its force a little weaker—in three-valued logic, Cpq is implied by ANpq, but does not imply it. The crucial case is that in which both p and q have the third truth-value, when CANpqCpq will be true but CCpqANpq neither true nor false. ($CAN\frac{1}{2}\frac{1}{2}C\frac{1}{2}\frac{1}{2} = CA\frac{1}{2}\frac{1}{2}C\frac{1}{2}\frac{1}{2} = C\frac{1}{2}1 = 1$; but $CC\frac{1}{2}\frac{1}{2}AN\frac{1}{2}\frac{1}{2} = C1\frac{1}{2} = \frac{1}{2}$). Similarly, Apq is not equivalent to CNpq but is a little stronger, implying but not being implied by it. To define Apq in terms of C, therefore, something a little stronger than CNpq is required. Or rather, we require something which is a little stronger than CNpq in three-valued logic, but which in two-valued logic is equivalent to it; for where the third truth-value is not involved, the tables for Apq and CNpq do coincide. This is a subtle problem, and its solution is ingenious.

One procedure which in general increases the logical force of an implicative statement is the weakening of its antecedent. Thus, 'Either Bartholomew or Philip will come' being a weaker assertion than 'Philip will come', 'If either Bartholomew or Philip comes I shall be surprised' is a stronger total assertion than 'If Philip comes I shall be surprised'. At the same time, there are cases in which this procedure will merely leave the force of the original implication unaltered. For example, the statement 'If either Bartholomew or Philip comes, Philip will come' is not really any stronger an assertion than 'If Bartholomew comes Philip will come', since it will be true in any case that Philip will come if Philip comes. Now one form of statement which is weaker than Np (the antecedent in our CNpq) is Cpq; for both in two-valued and in three-valued logic 'Not-p' implies 'If p then q' whatever q may be, but is not always implied by it. Hence the replacement of Np in CNpq by this weaker proposition Cpq will yield either a stronger assertion than the original CNpq or one equivalent to it; and it turns out to yield an equivalent form in two-valued logic and a stronger one in three-valued. In two-valued logic, the replacement of CNpq by CCpqq has something of the artificiality of the replacement of Cpq by CApqq in our example above, and makes no difference. (It does in fact amount to the replacement of CNpq by CANpqq, since in this logic Cpq is equivalent to ANpq). But in three-valued logic, when $p = \frac{1}{2}$ and $q = \frac{1}{2}$ CNpq and CCpqq will have different truth-values, the former being true ($CN\frac{1}{2}\frac{1}{2} = C\frac{1}{2}\frac{1}{2} = 1$) and the latter not ($CC\frac{1}{2}\frac{1}{2}\frac{1}{2} = C1\frac{1}{2} = \frac{1}{2}$); and this is precisely the point at which, in the three-valued system, the truth-tables for CNpq and Apq are different. CCpqq consequently serves ideally for the definition of Apq when this system is axiomatised.

The relative weakness of the relation represented in the three-valued system by 'C' may also be brought out by considering the force of the statement CNpp, 'If not-p then p'. In most senses of 'imply', when a proposition is implied even by its own negation, we may infer that the proposition in question is true. (This is the 'Law of Clavius', also referred to by certain Polish Jesuits as the *consequentia mirabilis*[6]). But the relative strength of the propositions 'If not-p then p' and the plain 'p' depends considerably upon the kind of implication involved. From 'Not-p strictly implies p' we may infer 'p', but not *vice versa* (we cannot infer 'Not-p strictly implies p' from 'p' but only from 'p is necessary'). But the weaker statement 'Not-p materially implies p' not only (in two-valued logic) implies the simple 'p' but is implied by it (for if 'p' is true it is materially implied by any proposition, including 'Not-p'). And with the still weaker CNpp of three-valued logic the direction of implication is reversed, CNpp in this sense being implied by p but not implying it. In either of the last two senses CNpp is true so long as Np is no closer to truth than p is; but whereas in two-valued logic the only way for this to happen is by Np being false and p true, in three-valued logic it may also happen by Np and p both having the value $\frac{1}{2}$.

When Np is no closer to truth than p is, whether because p is true and Np consequently false, or because p has the value $\frac{1}{2}$ and Np consequently the same, we might describe p as 'possible' (*möglich*). The assertion CNpp, abbreviated to 'Mp', is therefore sometimes read 'It is possible that p'. NMp will then be 'It is impossible that p'; and NMNp ('It is not possible that not-p') will be 'It is necessary that p'. We shall sometimes for convenience abbreviate 'NMp' to 'Ip' and 'NMNp' to 'Sp' ('S' being the symbol for 'It is necessary that' in some modal systems using the Polish notation[7]); and we shall also use 'Qp' for 'It is contingent that p', taking this to mean that both p and Np are possible, i.e. KMpMNp. The above tables for 'C' and 'N' yield the following values for Mp (CNpp) with different values of its argument:

$$\mathrm{M}1 = \mathrm{CN}11 = \mathrm{C}01 = 1$$
$$\mathrm{M}\tfrac{1}{2} = \mathrm{CN}\tfrac{1}{2}\tfrac{1}{2} = \mathrm{C}\tfrac{1}{2}\tfrac{1}{2} = 1$$
$$\mathrm{M}0 = \mathrm{CN}00 = \mathrm{C}10 = 0$$

The corresponding correlations for 'I', 'S' and 'Q' work out as follows: $\mathrm{I}1 = 0$, $\mathrm{I}\frac{1}{2} = 0$, $\mathrm{I}0 = 1$; $\mathrm{S}1 = 1$, $\mathrm{S}\frac{1}{2} = 0$, $\mathrm{S}0 = 0$; $\mathrm{Q}1 = 0$, $\mathrm{Q}\frac{1}{2} = 1$, $\mathrm{Q}0 = 0$.

Is this modal language really appropriate? No doubt our final decision about that must wait upon the interpretation we give to the system's three 'truth-values'; but let us first compare the purely formal properties of

[6]Cf. Lukasiewicz, *Aristotle's Syllogistic*, pp. 50-51, 80. Although this law 'If not-p implies p, then p', CCNppp, does not hold in the three-valued system, Wajsberg's axiom CCCpNppp will be seen in the light of what follows to assert something that is quite close to it, namely that if the very possibility of not-p implies p, then p.

[7]So, e.g., Bochenski in *La Logique de Théophraste* (Fribourg, 1947). In R. Feys's article 'Les Systèmes Formalisés des Modalités Aristotéliciennes' (*Revue Philosophique de Louvain*, Nov. 1950) the symbol 'L' is used; but we shall avoid this here as it suggests *logical* necessity, and we shall see that it is important to distinguish the necessity expressed by 'NMN' in this system from logical necessity.

'M', 'S', etc. with those of ordinary modal operators. It has recently been shown by von Wright[8] that the greater part of ordinary modal logic can be deduced from the ordinary laws of propositional logic together with the special modal distributive principle CMApqAMpMq (If 'Either p or q' is possible, then either p is possible or q is possible) and the consequence *ab esse ad posse*, CpMp (If p, then it is possible that p). When 'M' is defined as the three-valued CNpp, both of these special laws are easily verifiable by the truth-table method. And CpMp, in the sense of CpCNpp, follows immediately, by the substitution of Np for q, from the axiom CpCqp. (This is an interesting illustration of the way in which this system makes modal logic continuous with the logic of propositions). We may also establish by means of the truth-tables Lewis's two 'paradoxical' laws CSpC'qp (where C'qp, 'q strictly implies p', is defined as SCqp) and CIpC'pq; and the Aristotelian law of contingency, CQpQNp (If any proposition is contingent then so is its contradictory), which Lukasiewicz has lately subjected to a rather curious attack.[9] On the other hand, certain features of the modal truth-tables themselves seem a little peculiar, from the point of view of ordinary modal logic. Consider, for example, the correlation 'I0 = 1'. Is it really the case that 'It is impossible that p' is automatically true if p happens to be false? 'Q1 = 0' (i.e. 'It is contingent that p' is automatically false if p happens to be true), 'Q0 = 0', 'S1 = 1' and 'M0 = 0' are similarly startling. We shall find, however, that their oddity largely disappears if we relate the system to the problem which it was originally designed for handling—the problem of 'future contingents'. To this we may now turn.

The terms 'proposition' and 'true' are nowadays generally used in such a way that we cannot speak of the truth-value of a proposition as altering with the passage of time. This usage, however, has not always been the common one. Ancient and medieval usage was generally such that logicians could speak (as Aristotle did speak[10]) of 'Socrates is sitting down' as a 'proposition' which is 'true' at those times at which he *is* sitting down and false at those times at which he is not. And what is more important, Aristotle speaks[11] of some propositions about the future as being neither true nor false when they are uttered, on the ground that there is as yet no definite fact with which they can accord or conflict. Professor Broad[12] has spoken in this way of all propositions whatever that refer (as we loosely say) to the future; but Aristotle speaks thus only of propositions about such future events as are not already predetermined. That there are such events he is convinced, for otherwise 'there would be no need to deliberate or take trouble, on the supposition that if we should adopt a

[8] G. H. von Wright, *An Essay in Modal Logic* (Amsterdam, 1951) Appendix II.

[9] J. Lukasiewicz, 'On Variable Functors of Propositional Arguments', *Proceedings of the Royal Irish Academy*, 54A2 (1951), §1.

[10] *Categories*, 4a24ff.

[11] *De Interpretatione*, Ch. 9.

[12] *Scientific Thought*, Ch. II.

certain course, a certain result would follow, while, if we did not, the result would not follow '. And ' since propositions correspond with facts ', i.e. their truth or falsehood depends on their relation to facts, ' it is evident that when in future events there is a real alternative, and a potentiality in contrary directions, the corresponding affirmation and denial have the same character ', i.e. have a potentiality both of being true and of being false, but are not actually either. Aristotle is, I think, grappling with a genuine difficulty here. Is it really possible to hold at one and the same time (*a*) that whether or not there will be a sea-battle tomorrow is as yet genuinely undetermined, and (*b*) that it is already either definitely true or definitely false that a sea-battle will occur to-morrow ? (In other words, can there be ' propositions ', in the timeless sense in which ' proposition ' is currently used, about events of this sort ?) For what is the case already has passed out of the realm of alternative possibilities into the realm of what cannot be altered. ' When it is, that which is is-necessarily, and when it is not, that which is not necessarily is-not ', i.e. when a thing passes from the future into the present and so into the past, its chance of being otherwise has disappeared.

Lukasiewicz's three-valued logic is admirably adapted to the expression of this way of regarding statements about contingent future events. The value ' 1 ', of course, attaches to statements which are definitely true, either because they refer to timeless relations (e.g. ' $2 + 2 = 4$ ') or because that of which they speak has already come to pass or is already coming to pass, or because its coming to pass is already determined ; the value ' 0 ' to statements which are definitely false for analogous reasons ; and the value ' $\frac{1}{2}$ ' to statements about the undetermined future. Given this interpretation, there is a clear sense in which what is definitely false is always ' impossible ' ($I0 = 1$) and what is definitely true always ' necessary ' ($S1 = 1$). For we have definite truth and definite falsehood only when the possibility of turning out one way or the other which attaches to some future events is for one reason or another absent.

An interesting feature of the modal functions, to which Jordan draws attention, is that they never take the third truth-value. For ' It is possible that p ' is definitely true not only when p is definitely true but also when it is not yet either true or false ; ' It is impossible that p ' definitely false under both these conditions ; and similarly with the others. This peculiarity accords well enough with our intuitive notion of a ' possibility ' as that which is somehow real even when that of which it is a possibility is not yet so ; and it has the effect of giving a two-valued character to the modal part of the three-valued system. Thus although ApNp, ' Either p or not-p ' is not a law of this system, it is a law that any proposition either is or is not possible, AMpNMp. (For we have $AM1NM1 = A1N1 = A10 = 1$; $AM\frac{1}{2}NM\frac{1}{2} = A1N1 = A10 = 1$; and $AM0NM0 = A0N0 = A01 = 1$). We can also say that any proposition either is or is not true (or false, or indeterminate). This is not, indeed, the sort of fact about the propositions of a system which can be expressed in the system itself; but my point is that even if the 'meta-system' in which we do express it is itself three-valued, the question as to the truth, falsehood or indeterminacy of a proposition of the original system is a question as to *present* and therefore determinate fact, so that the logic or part of logic with which we handle such a question is itself in effect two-valued.

Aristotle's chapter on 'future contingents' was the subject of much discussion among the later medieval logicians, who were worried by the problem of reconciling Aristotle's views here (if this could be done) with the doctrine of God's foreknowledge[13]. In connection with the Aristotelian statement quoted above, that 'When it is, whatever is is-necessarily, and when it is not, whatever is not necessarily is-not', numerous medieval commentators (and some modern ones[14]) have argued that we cannot say that 'whatever is is-necessarily', but only that 'necessarily, whatever is is'. This criticism seems to assume that the necessity of which Aristotle here speaks is *logical* necessity. A thing's being does not make the proposition that it is a logically necessary proposition, though the complete proposition 'Whatever is is' is logically necessary. Aristotle was not blind to the distinction here made, for he makes it himself in other contexts,[15] and if we are correct in our surmise that the necessity to which he refers in 'When it is, etc.' is necessity of a different sort, the criticism is beside the point. It is in any case important to notice that logical necessity is not what the 'NMN' of Lukasiewicz's three-valued logic refers to. For 'NMNp' is in this system a truth-function, while 'It is logically necessary that p' is in no system a truth-function, but rather expresses a consequential higher-order characteristic of some truth-functions. For example, the assertion that 'If Socrates is dead he is dead' is logically necessary is not automatically made true by the fact that its argument, 'If Socrates is dead he is dead', has the truth-value it has, namely truth; it is true, rather, because the function 'If p then p', which 'If Socrates is dead he is dead' exemplifies, is true no matter what the truth-value of p may be. On the other hand 'NMN If-Socrates-is-dead-he-is-dead' (where NMN is interpreted as in the system now being considered) *is* true simply because 'If Socrates is dead he is dead' is true; that is, it is true for precisely the same sort of reason as 'NMN Socrates-is-dead' is now true.[16] Logically necessary propositions do of course form a sub-class of 'necessary' propositions in the sense of the system.

[13]See, e.g. P. Boehner's edition of Ockham's *Tractatus de Praedestinatione et de Praescientia Dei et de Futuris Contingentibus* (Franciscan Institute Publications No. 2, 1945). Boehner considers the relation of Aristotle's and Ockham's views to three-valued logic in his Introduction.

[14]e.g. Lewis, p. 215; C. A. Baylis, 'Are Some Propositions Neither True nor False?', *Philosophy of Science*, 1936, pp. 161-2.

[15]See especially *De Soph. Elench.* 116 a 24 ff.

[16]On the non-truth-functional character of logical modalities, I have sufficiently insisted elsewhere. I must, in fact, include myself among those who have, through concentrating too exclusively on the logical modalities, treated the possibility of truth-functional modalities with excessive scepticism.

The distinction between Lukasiewicz's truth-functional necessity and logical necessity may also be brought out by considering the following case : In Lukasiewicz's system, whenever Np is true we have not only NMp but also, and consequently, CNpNMp. (' If not-p then not possibly p ' is true under these conditions because its antecedent and consequent are true). And since CNpNMp is (in these circumstances) true, it is (in these circumstances) ' necessary '. But it is not for that reason a logical law. If it did turn out to be a logical law, CMpp would also be a logical law (for CMpp follows from CNpNMp by the substitution of Mp for q in the axiom CCNpNqCqp and detachment of the consequent). And if CMpp were a law, since in any case CpMp is a law, ' p ' and ' Mp ' would be mutually inferable, the distinction between truth and indeterminacy would disappear (for Mp never takes the third truth-value) and the three-valued logic would collapse into a two-valued one. But in fact, although CNpNMp is true when p is true as well as when p is false, it is not true when p is indeterminate ; for we then have $CN\frac{1}{2}NM\frac{1}{2} = CN\frac{1}{2}N1 = C\frac{1}{2}0 = \frac{1}{2}$. Hence it is not true regardless of the truth-value of its arguments, *i.e.* it is not a logical law ; so the system stands firm[17].

We may contrast this with another case in which ' It is impossible that p ' is implied by what appears to be a weaker proposition, and in which the implication *is* a logical law. The apparently weaker proposition is ' It is *possibly* impossible that p '—we do have, for all values of p, CMNMpNMp, ' If it is possible that p is not possible, then p is not possible '. It is only possible for p to be impossible when it is true that p is impossible ; for although ' p is impossible ' might also be possible if it were indeterminate, indeterminacy is in fact a truth-value which NMp, being a modal function, never has. This thesis CMNMpNMp, von Wright has pointed out[18], is a distinguishing law of Lewis's ' strongest ' modal system S5, though there are other modal systems in which it does not hold.

In sum, three-valued logic does seem to bring new precision to our handling of statements with tenses (as opposed to the fundamentally tenseless propositions of the common systems) ; and we may say that Lukasiewicz has, by means of it, done for Aristotle's chapter on ' future contingents ' what he has done elsewhere for the Aristotelian theory of the syllogism. This does not mean, however, in this case any more than in the other, that in being given this new form the substance of the Aristotelian position survives without alteration. There is at least one feature of the Aristotelian account of future contingents which a three-valued logic seems incapable of preserving. For Aristotle held not only that (1) if p is a proposition about contingent future events (e.g. ' There will be a sea-battle to-morrow '), it is neither true nor false ; but also that (2) the disjunctive proposition

[17]Lukasiewicz introduces this proposition CNpNMp, and points out the consequences of supposing it a logical law, when discussing the Aristotelian ' When it is not, whatever is not necessarily is-not ' ; and there has been considerable argument about it. But the above seem to be the plain facts of the matter.

[18]*Op. cit.* See also his ' Interpretations of Modal Logic ', *Mind*, Apr. 1952.

' Either p or not p ' (' Either there will or there will not be a sea-battle to-morrow '), being not a contingent but a necessary disjunction, is always true. But, as we have already noted, ApNp is not one of the laws of the Lukasiewicz-Tarski three-valued system—ApNp = 1 when p = 0 or 1, but when p = $\frac{1}{2}$, ApNp = A$\frac{1}{2}$N$\frac{1}{2}$ = A$\frac{1}{2}\frac{1}{2}$ = $\frac{1}{2}$. Would Aristotle, perhaps, have defended his position by so using ' Either ' that a disjunction of indeterminate propositions is not itself automatically indeterminate, but automatically true ? Hardly. It is plain, I think, that Aristotle would not have regarded a disjunction of indeterminate propositions as ' automatically ' anything—he would have said that usually A$\frac{1}{2}\frac{1}{2}$ = $\frac{1}{2}$, but if the ' q ' in ' Apq happens to be ' Not p ', the disjunction is not indeterminate but true. This amounts to saying that in the three-valued logic of Aristotle, so far as he had such a thing, disjunction was not a truth-function. Or alternatively we may say—and this, I think, is the simple truth—that at this point Aristotle was quite excusably muddled, and was trying to use ' proposition ', ' true ', etc., at once in senses in which the logic of these things is two-valued and in senses in which it is three-valued.

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