X.—NOTES.

A LOGICAL PARADOX.

By Lewis Carroll.

"What, *nothing* to do?" said Uncle Jim. "Then come along with me down to Allen's. And you can just take a turn while I get myself shaved."

"All right," said Uncle Joe. "And the Cub had better come too, I suppose?"

The "Cub" was *me*, as the reader will perhaps have guessed for himself. I'm turned *fifteen*—more than three months ago; but there's no sort of use in mentioning *that* to Uncle Joe: he'd only say "Go to your cubbicle, little boy!" or "Then I suppose you can do cubbic equations?" or some equally vile pun. He asked me yesterday to give him an instance of a Proposition in *A*. And I said "All uncles make vile puns". And I don't think he liked it. However, that's neither here nor there. I was glad enough to go. I *do* love hearing those uncles of mine "chop logic," as they call it; and they're desperate hands at it, *I* can tell you!

"That is not a logical inference from my remark," said Uncle Jim.

"Never said it was," said Uncle Joe: "it's a *Reductio ad Absurdum*".

"An *Illicit Process of the Minor!*" chuckled Uncle Jim.

That's the sort of way they always go on, whenever *I'm* with them. As if there was any fun in calling me a Minor!

After a bit, Uncle Jim began again, just as we came in sight of the barber's. "I only hope *Carr* will be at home," he said. "Brown's so clumsy. And Allen's hand has been shaky ever since he had that fever."

"Carr's *certain* to be in," said Uncle Joe.

"I'll bet you sixpence he *isn't!*" said I.

"Keep your bets for your betters," said Uncle Joe. "I mean"—he hurried on, seeing by the grin on my face what a slip he'd made—"I mean that I can *prove* it, logically. It isn't a matter of *chance*."

"Prove it *logically!*" sneered Uncle Jim. "Fire away, then! I defy you to do it!"

"For the sake of argument," Uncle Joe began, "let us assume Carr to be *out*. And let us see what that assumption would lead to. I'm going to do this by *Reductio ad Absurdum*."

"Of course you are!" growled Uncle Jim. "Never knew any argument of *yours* that didn't end in some absurdity or other!"

"Unprovoked by your unmanly taunts," said Uncle Joe in a lofty tone, "I proceed. Carr being out, you will grant that, if Allen is *also* out, *Brown* must be at home?"

"What's the good of *his* being at home?" said Uncle Jim. "I don't want *Brown* to shave me! He's too clumsy."

"Patience is one of those inestimable qualities——" Uncle Joe was beginning; but Uncle Jim cut him off short.

"*Argue!*" he said. "Don't *moralise!*"

"Well, but *do* you grant it?" Uncle Joe persisted. "Do you grant me that, if Carr is out, it follows that if Allen is out Brown *must* be in?"

"Of course he must," said Uncle Jim; "or there'd be nobody to mind the shop."

"We see, then, that the absence of Carr brings into play a certain

Hypothetical, whose *protasis* is 'Allen is out,' and whose *apodosis* is 'Brown is in'. And we see that, so long as Carr remains out, this Hypothetical remains in force?"

"Well, suppose it does. What then?" said Uncle Jim.

"You will also grant me that the truth of a Hypothetical—I mean its *validity* as a logical *sequence*—does not in the least depend on its *protasis* being actually *true*, nor even on its being *possible*. The Hypothetical 'If you were to run from here to London in five minutes you would surprise people,' remains true as a *sequence*, whether you can do it or not."

"I *ca'n't* do it," said Uncle Jim.

"We have now to consider *another* Hypothetical. What was that you told me yesterday about Allen?"

"I told you," said Uncle Jim, "that ever since he had that fever he's been so nervous about going out alone, he always takes Brown with him."

"Just so," said Uncle Joe. "Then the Hypothetical 'if Allen is out Brown is out' is *always* in force, isn't it?"

"I suppose so," said Uncle Jim. (He seemed to be getting a little nervous, himself, now.)

"Then, if Carr is out, we have *two* Hypotheticals, 'if Allen is out Brown is *in*' and 'If Allen is out Brown is *out*,' in force at once. And two *incompatible* Hypotheticals, mark you! They ca'n't *possibly* be true together!"

"*Ca'n't* they?" said Uncle Jim.

"How *can* they?" said Uncle Joe. "How *can* one and the same *protasis* prove two contradictory *apodoses?* You grant that the two *apodoses*, 'Brown is *in*' and 'Brown is *out*,' *are* contradictory, I suppose?"

"Yes, I grant *that*," said Uncle Jim.

"Then I may sum up," said Uncle Joe. "If Carr is out, these two Hypotheticals are true together. And we know that they *cannot* be true together. Which is absurd. Therefore Carr *cannot* be out. There's a nice *Reductio ad Absurdum* for you!"

Uncle Jim looked thoroughly puzzled: but after a bit he plucked up courage, and began again. "I don't feel at all clear about that *incompatibility*. Why shouldn't those two Hypotheticals be true together? It seems to me that would simply prove '*Allen* is in'. Of course it's clear that the *apodoses* of those two Hypotheticals are incompatible—'Brown is in' and 'Brown is out'. But why shouldn't we put it like this? If Allen is out Brown is *out*. If Carr and Allen are *both* out, Brown is *in*. Which is absurd. Therefore Carr and Allen ca'n't be *both* of them out. But, so long as Allen is *in*, I don't see what's to hinder Carr from going *out*."

"My dear, but most illogical, brother!" said Uncle Joe. (Whenever Uncle Joe begins to "dear" you, you may make pretty sure he's got you in a cleft stick!) "Don't you see that you are wrongly dividing the *protasis* and the *apodosis* of that Hypothetical? Its *protasis* is simply 'Carr is out'; and its *apodosis* is a sort of sub-Hypothetical, 'If Allen is out, Brown is *in*'. And a most absurd apodosis it is, being hopelessly incompatible with that other Hypothetical, that we know is *always* true, 'If Allen is out, Brown is *out*'. And it's simply the assumption 'Carr is out' that has caused this absurdity. So there's only *one* possible conclusion. *Carr is in!*"

How long this argument *might* have lasted, I haven't the least idea. I believe *either* of them could argue for six hours at a stretch. But, just at this moment, we arrived at the barber's shop; and, on going inside, we found——

Note.

The paradox, of which the foregoing paper is an ornamental presentment, is, I have reason to believe, a very real difficulty in the Theory of Hypotheticals. The disputed point has been for some time under discussion by several practised logicians, to whom I have submitted it; and the various and conflicting opinions, which my correspondence with them has elicited, convince me that the subject needs further consideration, in order that logical teachers and writers may come to some agreement as to what Hypotheticals *are*, and how they ought to be treated.

The original dispute, which arose, more than a year ago, between two students of Logic, may be symbolically represented as follows:—

There are two Propositions, *A* and *B*.

It is given that

(1) If *C* is true, then, if *A* is true, *B* is not true;

(2) If *A* is true, *B* is true.

The question is, can *C* be true?

The reader will see that if, in these two Propositions, we replace the letters *A*, *B*, *C* by the names Allen, Brown, Carr, and the words "true" and "not true" by the words "out" and "in" we get

(1) If Carr is out, then, if Allen is out, Brown is in;

(2) If Allen is out, Brown is out.

These are the very two Propositions on which "Uncle Joe" builds his argument.

Several very interesting questions suggest themselves in connexion with this point, such as

Can a Hypothetical, whose protasis is false, be regarded as legitimate?

Are two Hypotheticals, of the forms "If *A* then *B*" and "If *A* then not-*B*," compatible?

What difference in meaning, if any, exists between the following Propositions?

(1) *A*, *B*, *C*, cannot be all true at once;

(2) If *C* and *A* are true, *B* is not true;

(3) If *C* is true, then, if *A* is true, *B* is not true;

(4) If *A* is true, then, if *C* is true, *B* is not true.

The following concrete form of the paradox has just been sent me, and may perhaps, as embodying *necessary* truth, throw fresh light on the question.

Let there be three lines, *KL*, *LM*, *MN*, forming, at *L* and *M*, equal acute angles on the same side of *LM*.

Let "*A*" mean "The points *K* and *N* coincide, so that the three lines form a triangle".

Let "*B*" mean "The triangle has equal base-angles".

Let "*C*" mean "The lines *KL* and *MN* are unequal".

Then we have

(1) If *C* is true, then, if *A* is true, *B* is not true;

(2) If *A* is true, *B* is true.

The second of these Propositions needs no proof; and the first is proved in Euc., i, 6, though of course it may be questioned whether it fairly represents Euclid's meaning.

I greatly hope that some of the readers of MIND who take an interest in logic will assist in clearing up these curious difficulties.

THE PERCEPTION OF DISTANCE IN THE INVERTED LANDSCAPE.[1]

In his discussion of the third dimension (vol. ii. p. 213 of the *Principles of Psychology*) Prof. James notes the fact that when a landscape is looked at with the head inverted, there is a remarkable increase of the

[1] From the Laboratory of the Cornell University, Ithaca, N.Y.

apparent distance of the horizon-line. He seems to regard this phenomenon as in some way supporting the nativistic view of the "depth sensation," and in a footnote observes: "What may be the physiological process connected with this increased sensation of depth is hard to discover. It seems to have nothing to do with the part of the retina affected, since the mere inversion of the picture (by mirrors reflecting prisms, &c.) without inverting the head does not seem to bring it about; nothing with sympathetic axial rotation of the eyes, which might enhance the perspective through exaggerated disparity of the two retinal images, for one-eyed persons get it as strongly as those with two eyes. I cannot find it to be connected with any alteration in the pupil or with any ascertainable strain in the muscles of the eyes, sympathising with those of the body." He adds at the end of the note: "I cannot help thinking that any one who can explain the exaggeration of the depth-sensation in this case will at the same time throw much light on its normal constitution". It would be interesting if Prof. James would publish a more detailed account of the experiments which led him to reject the explanations he mentions in this note. A full description of the experiments when he found that "the mere inversion of the picture by mirrors reflecting prisms, &c.," does not produce the effect in question, would be especially valuable. For it seems at least possible, *a priori*, that the "part of the retina affected" may have some influence upon the estimation of distance—and in the following way. It is a well-known though unexplained fact that the height of the upper half of the field of sight is over-estimated, while that of the lower half is under-estimated (see Wundt, *Phys. Psych.*, ii. 121). The example of the inverted S is familiar. Now, if we suppose a schematic landscape representing a comparatively level foreground stretching away to a horizon-line which divides the field of sight into equal parts and above which appear mountains or other elevated objects, then when the head is in a normal position the vertical dimension of the foreground, which occupies the lower half of the field of sight, will be under-estimated. On the contrary, that of the objects at the horizon, and of the sky above, will be over-estimated. But when the head is inverted the foreground, extending to the horizon-line, will fall in the upper half of the visual field and be over-estimated, while all distant objects will be under-estimated—mountains will seem lower, &c. Size being a criterion of the distance of known objects, this latter effect, combined with the apparent lengthening of the foreground, might easily produce an "increase of the depth-sensation". Of course, an ordinary landscape presents irregularities which would greatly affect the working of this principle.

To test the explanation just stated, a few preliminary experiments were made with the help of mirrors placed at such an angle that the erect and inverted images of the view—a rather extensive one—from the laboratory window might be looked at side by side. Four persons found that, in opposition to Prof. James's results, an increase of horizon-distance was evident in the inverted image; and one of these observers, entirely unconscious of the theory at stake, judged the distance to be greater on transferring attention from the erected to the inverted picture, from the fact that the far-off hills appeared to have decreased in *size*.

More complete results were later obtained by the use of stereoscopic views, which were shown first erect and then inverted. Here again the subjects not only contradicted Prof. James's statement, that apparent recession of the horizon does not occur under these circumstances, but furnished evidence in support of the explanation offered in this paper.

The first picture examined, a view of the Aar glacier, approached very closely to the schematic landscape described above. It represents a nearly level field of ice stretching away to mountains, the base-line of which divides foreground from background at about one-half the height of the picture. Out of ten observers, seven noticed a recession of the horizon when the photograph was inverted; two were doubtful, and one said that the tops of the mountains appeared nearer than before—that is, the mountains and sky in the reversed picture seemed to slope towards the observer as an ordinary foreground would. Out of the seven persons who noticed the effect of increased depth, six, on being questioned as to any change in the apparent height of the part of the photograph representing the foreground, said that it seemed slightly greater when occupying the upper half of the visual field. One suggested this as an explanation of the illusion of increased distance.

A second photograph experimented with is of a scene on the Killarney Lakes. The shore-line falls at about two-thirds the height of the picture, measuring from the bottom. Mountains nearly fill the uppermost third. At half the height of the picture a point of land projects entirely across, the lake appearing above and below it. Evidently the conditions are more complicated here, and we should expect less definite results. To five out of ten observers the mountains seemed farther away when the picture was inverted; two said that the distance of the projecting point was increased, and three were doubtful, or thought the distance of the mountains diminished. Among the first-mentioned five, two said that the height of the picture from the base-line to the point was increased by inverting the picture—that is, they noticed the tendency to overestimate the upper half of the visual field; two said that the stretch of water above the point and the mountains at the top seemed shorter when the photograph was reversed—that is, they noticed the tendency to under-estimate the lower half of the field of sight. As for the two observers who found an increase in the distance of the point, one declared the height of the foreground to have increased, while the other said that the height of the background had diminished.

Finally, a view of Heidelberg and the Neckar afforded a tolerably satisfactory "negative instance". Here the horizon is very distant, low-lying and faint, and the horizon-line is at about two-thirds the height of the picture. There is no immediate foreground, the photograph having been taken from a height above the town. The lower part of the photograph is occupied by houses which are at a considerable distance from the point of view. There is nothing whatever in the picture to divide the upper from the lower half of the visual field. Eight persons out of ten found that the horizon-line seemed *nearer* when the photograph was upside down, and two noticed no change. The illusion of an approach of the horizon is easily explained. The uppermost third of the picture represents an extent of sky. When the picture was inverted this expanse irresistibly suggested a foreground of water, and owing to its comparative narrowness, and to the fact that its width was underestimated, as soon as it was brought into the lower half of the field the horizon-line which bounded it looked much nearer than before. In the other pictures, the illusion of water did not occur, because the sky is there bounded by the curved lines of mountain-tops.

A less methodical examination of several other photographs afforded a general confirmation of these results.

Absolutely conclusive experiments on the point in question are difficult to devise, but the results just stated certainly do not disprove the theory that an error in the estimation of size may at least partly cause the observed "increase of the depth-sensation".

MARGARET WASHBURN.