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A FUNCTIONAL CALCULUS OF FIRST ORDER BASED ON STRICT IMPLICATION

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The following system[1] is an extension of the Lewis calculus S2 to include quantification.[2]

Primitive symbols.

( ) {parentheses}.

$\sim$ {negation}.

$\Diamond$ {modal operator}.

$\cdot$ {conjunction}.

($\exists$) {existential quantifier, the blank space replaced by an appropriate variable}.

Propositional variables $p, q, r, s, t, p_1, q_1, r_1, s_1, t_1, \cdots, p_n, q_n, r_n, s_n, t_n, \cdots$.

Functional variables of degree $1, 2, \cdots, n, \cdots$: monadic functional variables $F_1, G_1, H_1, F_1^1, \cdots$; dyadic functional variables $F_2, G_2, H_2, F_2^1, \cdots$; n-adic functional variables $F_n, G_n, H_n, F_n^1 \cdots; \cdots$.

Formula is defined as any finite sequence of primitive symbols.

Syntactic notation. Greek letters will be used as variables in the syntax language. Upper case Greek letters $\mathrm{A}, \mathrm{B}, \Gamma, \mathrm{E}, \mathrm{H}, \mathrm{A}_1, \mathrm{B}_1, \Gamma_1, \mathrm{E}_1, \mathrm{H}_1, \cdots, \mathrm{A}_n, \mathrm{B}_n, \Gamma_n, \mathrm{E}_n, \mathrm{H}_n$ represent formulas. Lower case Greek letters $\alpha, \beta, \gamma, \delta, \alpha_1, \beta_1, \gamma_1, \delta_1, \cdots, \alpha_n, \beta_n, \gamma_n, \delta_n$ represent individual variables unless otherwise specified.

Well-formed formula. Write "wff" for "well-formed formula" and "wf" for "well-formed." A wff is defined recursively as follows;

A propositional variable is wf.

$\beta(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is wf where β is an n-adic functional variable and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are individual variables.

If A and B are wff's then $\sim\mathrm{A}$, $\Diamond\mathrm{A}$, $(\mathrm{A}\cdot\mathrm{B})$, $(\exists\alpha)\mathrm{A}$ are wff's.

The only wff's are those which follow from this definition.

Capital Greek letters will hereafter be restricted to the representation of wff's.

Bound and free individual variables. An occurrence of an individual variable α in a wff A is a bound occurrence if it is in a wf part of A of the form $(\exists\alpha)\mathrm{B}$. Otherwise it is a free occurrence.

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The method for constructing this system was suggested in part by Alonzo Church's ***Introduction to mathematical logic,*** Princeton University Press, 1944.

[2] Lewis and Langford, ***Symbolic logic,*** The Century Co., 1932.

Definition schemata. $=_{df}$ between two expressions indicates that the formula on the left abbreviates the formula on the right.

$(\alpha)A =_{df} \sim(\exists\alpha)\sim A.$
$(A \vee B) =_{df} \sim(\sim A \cdot \sim B).$
$(A \prec B) =_{df} \sim\Diamond(A \cdot \sim B).$
$(A = B) =_{df} ((A \prec B)\cdot(B \prec A)).$
$(A \supset B) =_{df} \sim(A \cdot \sim B).$
$(A \equiv B) =_{df} ((A \supset B)\cdot(B \supset A)).$

The outermost parentheses will be omitted. The dot for conjunction will also be omitted wherever unambiguous. Rules will be distinguished by Roman numerals.

Axiom schemata.

1. $(AB) \prec (BA).$
2. $(AB) \prec A.$
3. $A \prec (AA).$
4. $(AB)\Gamma \prec A(B\Gamma).$
5. $((A \prec B)(B \prec \Gamma)) \prec (A \prec \Gamma).$
6. $(A(A \prec B)) \prec B.$
7. $\Diamond(AB) \prec \Diamond A.$
8. $(\alpha)A \prec B$, where α and β are individual variables, no free occurrence of α in A is in a wf'd part of A of the form $(\beta)\Gamma$ and B results from the substitution of β for all free occurrences of α in A.
9. $(\alpha)(A \supset B) \prec ((\alpha)A \supset (\alpha)B).$
10. $A \prec (\alpha)A$, where α is not free in A.
11. $\Diamond(\exists\alpha)A \prec (\exists\alpha)\Diamond A.$

Rules of inference.

I. From A and $A \prec B$ infer B (modus ponens, abbreviate: mod pon).
II. From A and B infer AB (adjunction, abbreviate: adj).
III. If A, B, and Γ are such that E results from Γ by the substitution of B for one or more occurrences of A in Γ, and if $A = B$, then infer Γ from E and E from Γ (substitution, abbreviate: subst).
IV. If B is the result of substituting the individual variable β for all free occurrences of α in A then infer $(\beta)B$ from A. (generalization, abbreviate: gen)

Proof of a wff A is a finite list of wff's $A_1, A_2, \cdots, A_n$ such that each A_i is an axiom or derivable from the preceding formulas of the list by applying any of the rules of inference. A is provable if it is the A_n of such a list.

"A is provable" will be abbreviated: $\vdash$A.

Correspondence with Lewis system S2. Axiom schemata 1–7 correspond to B1–B4, B6–B8.[3] B5 is derivable as shown by McKinsey.[4] Rules I, II, and

[3] See Lewis and Langford, p. 493.

[4] J. C. C. McKinsey, *Bulletin of the American Mathematical Society,* vol. 40 (1934), pp. 425–427.

III correspond to Lewis's rules of inference, adjunction and substitution (a).[5] We can dispense with substitution (b) since we have an infinite list of axioms.

The following notational changes will be assumed as made when referring to the theorems of *Symbolic logic*: Lewis's propositional variables replaced by capital Greek letters, and $=$ replaced by $\equiv$ or $=_{df}$ according as an equivalence or a defining relation is indicated, and dots for brackets and brackets replaced by parentheses.

When reference is made in the proofs to a theorem in *Symbolic logic* the number will be italicized. Otherwise the numbers refer to the list of axioms and theorems developed here.

Some rules useful for facilitating proofs.

V. If $\vdash A \prec B$ then $\vdash A \equiv (AB)$.

$\vdash (A \prec B) \equiv (A \prec (AB))$ *16.33*

$\vdash A \prec B$ hyp

$\vdash A \prec (AB)$ subst

$\vdash (AB) \prec A$ 2

$\vdash (A \prec (AB))((AB) \prec A)$ adj

$\vdash A \equiv (AB)$ def

VI. If $\vdash A \prec B$ then $\vdash \Diamond A \prec \Diamond B$.

$\vdash A \prec B$ hyp

$\vdash A \equiv (AB)$ V

$\vdash \Diamond(AB) \prec \Diamond B$ *19.13*

$\vdash \Diamond A \prec \Diamond B$ subst

Assume that each step in the following proofs is preceded by $\vdash$.

VII. If $\vdash A \prec B$ then $\vdash \sim\Diamond\sim A \prec \sim\Diamond\sim B$.

$A \prec B$ hyp

$A \equiv (AB)$ V

$\sim\Diamond\sim(AB) \prec \sim\Diamond\sim B$ *19.24*

$\sim\Diamond\sim A \prec \sim\Diamond\sim B$ subst

VIII. If $\vdash A_1 \prec A_2$, $\vdash A_2 \prec A_3$, $\cdots$, $\vdash A_{n-1} \prec A_n$ then $\vdash A_1 \prec A_n$.

$(A_1 \prec A_2)(A_2 \prec A_3)$ hyp, adj

$((A_1 \prec A_2)(A_2 \prec A_3)) \prec (A_1 \prec A_3)$ 5

$A_1 \prec A_3$ mod pon

$(A_1 \prec A_3)(A_3 \prec A_4)$ hyp, adj

$((A_1 \prec A_3)(A_3 \prec A_4)) \prec (A_1 \prec A_4)$ 5

$A_1 \prec A_4$ mod pon

$\vdots$

$A_1 \prec A_n$ mod pon

[5] See Lewis and Langford, p. 125.

IX. If $\vdash A \mathrel{-\!\!3} B$ and $\vdash \Gamma \mathrel{-\!\!3} E$ then $\vdash (A\Gamma) \mathrel{-\!\!3} (BE)$.

$((A \mathrel{-\!\!3} B)(\Gamma \mathrel{-\!\!3} E)) \mathrel{-\!\!3} ((A\Gamma) \mathrel{-\!\!3} (BE))$ *19.68*

$(A \mathrel{-\!\!3} B)(\Gamma \mathrel{-\!\!3} E)$ hyp, adj

$(A\Gamma) \mathrel{-\!\!3} (BE)$ mod pon

X. If $\vdash A \mathrel{-\!\!3} B$ then $\vdash \sim B \mathrel{-\!\!3} \sim A$.

$A \mathrel{-\!\!3} B$ hyp

$(A \mathrel{-\!\!3} B) \mathrel{-\!\!3} (\sim B \mathrel{-\!\!3} \sim A)$ *12.43*

$\sim B \mathrel{-\!\!3} \sim A$ mod pon

XI. If $\vdash A \equiv B$ then $\vdash \sim A \equiv \sim B$.

$((A \mathrel{-\!\!3} B)(B \mathrel{-\!\!3} A)) \mathrel{-\!\!3} ((A \mathrel{-\!\!3} B)(B \mathrel{-\!\!3} A))$ *12.1*

$((A \mathrel{-\!\!3} B)(B \mathrel{-\!\!3} A)) \mathrel{-\!\!3} ((\sim B \mathrel{-\!\!3} A)(\sim A \mathrel{-\!\!3} \sim B))$ *12.44*, subst

$(A \mathrel{-\!\!3} B)(B \mathrel{-\!\!3} A)$ hyp, def

$(\sim B \mathrel{-\!\!3} \sim A)(\sim A \mathrel{-\!\!3} \sim B)$ mod pon

$\sim A \equiv \sim B$ *12.15*, subst, def

XII. If $\vdash A \mathrel{-\!\!3} B$ and $\vdash A \mathrel{-\!\!3} \Gamma$ then $\vdash A \mathrel{-\!\!3} (B\Gamma)$.

$((A \mathrel{-\!\!3} B)(A \mathrel{-\!\!3} \Gamma)) \mathrel{-\!\!3} (A \mathrel{-\!\!3} (B\Gamma))$ *19.61*

$(A \mathrel{-\!\!3} B)(A \mathrel{-\!\!3} \Gamma)$ hyp, adj

$(A \mathrel{-\!\!3} (B\Gamma))$ mod pon

13. $\vdash (\exists\alpha)A \equiv \sim(\alpha)\sim A$.

$\sim\sim(\exists\alpha)A \equiv \sim(\alpha)\sim A$ *12.1*, def, *12.3*, subst, XI

$(\exists\alpha)A \equiv \sim(\alpha)\sim A$ *12.3*, subst

14. $\vdash (\exists\alpha)\sim A \equiv \sim(\alpha)A$.

$(\exists\alpha)\sim A \equiv \sim(\alpha)\sim\sim A$ 13

$(\exists\alpha)\sim A \equiv \sim(\alpha)A$ *12.3*, subst

15. $\vdash \sim(\exists\alpha)A \equiv (\alpha)\sim A$.

$(\exists\alpha)A \equiv \sim(\alpha)\sim A$ 13

$\sim(\exists\alpha)A \equiv \sim\sim(\alpha)\sim A$ XI

$\sim(\exists\alpha)A \equiv (\alpha)\sim A$ *12.3*, subst

16. $\vdash B \mathrel{-\!\!3} (\exists\alpha)A$, where B, A, and α are as in 8.

$(\alpha)\sim A \mathrel{-\!\!3} \sim B$ 8

$\sim\sim B \mathrel{-\!\!3} \sim(\alpha)\sim A$ X

$B \mathrel{-\!\!3} \sim(\alpha)\sim A$ *12.3*, subst

$B \mathrel{-\!\!3} (\exists\alpha)A$ 13, subst

17. $\vdash (\alpha)A \mathrel{-\!\!3} (\exists\alpha)A$.

$(\alpha)A \mathrel{-\!\!3} A$ 8

$A \mathrel{-\!\!3} (\exists\alpha)A$ 16

$(\alpha)A \mathrel{-\!\!3} (\exists\alpha)A$ VIII

18. $\vdash (\alpha)\sim\Diamond\sim A \prec \sim\Diamond\sim(\alpha)A.$

$\Diamond(\exists\alpha)\sim A \prec (\exists\alpha)\Diamond\sim A$ 11

$\sim(\exists\alpha)\Diamond\sim A \prec \sim\Diamond(\exists\alpha)\sim A$ X

$(\alpha)\sim\Diamond\sim A \prec \sim\Diamond(\exists\alpha)\sim A$ 15, subst

$(\alpha)\sim\Diamond\sim A \prec \sim\Diamond\sim(\alpha)\sim\sim A$ 13, subst

$(\alpha)\sim\Diamond\sim A \prec \sim\Diamond\sim(\alpha)A$ *12.3*, subst

19. $\vdash (\alpha)(A \prec B) \prec ((\alpha)A \prec (\alpha)B).$

$(\alpha)(A \supset B) \prec ((\alpha)A \supset (\alpha)B)$ 9

$\sim\Diamond\sim(\alpha)(A \supset B) \prec \sim\Diamond\sim((\alpha)A \supset (\alpha)B)$ VII

$(\alpha)\sim\Diamond\sim(A \supset B) \prec \sim\Diamond\sim(\alpha)(A \supset B)$ 18

$(\alpha)\sim\Diamond\sim(A \supset B) \prec \sim\Diamond\sim((\alpha)A \supset (\alpha)B)$ VIII

$(\alpha)(A \prec B) \prec ((\alpha)A \prec (\alpha)B)$ *18.7*, subst

20. $\vdash A \equiv (\alpha)A$, where α is not free in A.

$A \prec (\alpha)A$ 10

$(\alpha)A \prec A$ 8

$(A \prec (\alpha)A)((\alpha)A \prec A)$ adj

$A \equiv (\alpha)A$ def

21. $\vdash A \equiv (\exists\alpha)A$, where α is not free in A.

$\sim A \equiv (\alpha)\sim A$ 20, hyp

$\sim\sim A \equiv \sim(\alpha)\sim A$ XI

$A \equiv (\exists\alpha)A$ *12.3*, subst, 13

22. $\vdash (\alpha)(A \prec B) \prec (A \prec (\alpha)B)$, where α is not free in A.

$(\alpha)(A \prec B) \prec ((\alpha)A \prec (\alpha)B)$ 19

$(\alpha)(A \prec B) \prec (A \prec (\alpha)B)$ 20, subst

23. $\vdash (\alpha)(A \prec B) \prec ((\exists\alpha)A \prec B)$, where α is not free in B.

$(\alpha)(A \prec B) \prec (\alpha)(A \prec B)$ *12.1*

$(\alpha)(A \prec B) \prec (\alpha)(\sim B \prec \sim A)$ *12.44*, subst

$(\alpha)(\sim B \prec \sim A) \prec (\sim B \prec (\alpha)\sim A)$ 22

$(\alpha)(A \prec B) \prec (\sim B \prec (\alpha)\sim A)$ VIII

$(\alpha)(A \prec B) \prec (\sim(\alpha)\sim A \prec \sim\sim B)$ *12.44*, subst

$(\alpha)(A \prec B) \prec ((\exists\alpha)A \prec B)$ 13, *12.3*, subst

XIII. If $\vdash A \prec B$ then $\vdash (\alpha)A \prec (\alpha)B$.

$A \prec B$ hyp

$(\alpha)(A \prec B)$ gen

$(\alpha)(A \prec B) \prec ((\alpha)A \prec (\alpha)B)$ 19

$(\alpha)A \prec (\alpha)B$ mod pon

24. $\vdash (\alpha)(A \equiv B) \prec (\alpha)A \equiv (\alpha)B).$

$((A \prec B)(B \prec A)) \prec (A \prec B)$ 2

$((A \prec B)(B \prec A)) \prec (B \prec A)$ *12.17*

$(\alpha)((A \prec B)(B \prec A)) \prec (\alpha)(A \prec B)$ XIII

$(\alpha)((A \prec B)(B \prec A)) \prec (\alpha)(B \prec A)$ XIII

$(\alpha)(A \strictif B) \strictif ((\alpha)A \strictif (\alpha)B)$ 19
$(\alpha)(B \strictif A) \strictif ((\alpha)B \strictif (\alpha)A)$ 19
$(\alpha)((A \strictif B)(B \strictif A)) \strictif ((\alpha)A \strictif (\alpha)B)$ VIII
$(\alpha)((A \strictif B)(B \strictif A)) \strictif ((\alpha)B \strictif (\alpha)A)$ VIII
$(\alpha)((A \strictif B)(B \strictif A)) \strictif (((\alpha)A \strictif (\alpha)B)((\alpha)B \strictif (\alpha)A))$ XII
$(\alpha)(A \equiv B) \strictif ((\alpha)A \equiv (\alpha)B)$ def

XIV. If $\vdash A \equiv B$ then $\vdash (\alpha)A \equiv (\alpha)B$.
$A \equiv B$ hyp
$(\alpha)A \equiv (\alpha)A$ *12.11*
$(\alpha)A \equiv (\alpha)B$ subst

25. $\vdash (\alpha)(A \strictif B) \strictif ((\exists\alpha)A \strictif (\exists\alpha)B)$.
$(\alpha)(A \strictif B) \strictif (\alpha)(A \strictif B)$ *12.1*
$(\alpha)(A \strictif B) \strictif (\alpha)(\sim B \strictif \sim A)$ *12.44*, subst
$(\alpha)(\sim B \strictif \sim A) \strictif ((\alpha)\sim B \strictif (\alpha)\sim A)$ 19
$((\alpha)\sim B \strictif (\alpha)\sim A) \strictif (\sim(\alpha)\sim A \strictif \sim(\alpha)\sim B$ *12.43*
$(\alpha)(A \strictif B) \strictif (\sim(\alpha)\sim A \strictif \sim(\alpha)\sim B)$ VIII
$(\alpha)(A \strictif B) \strictif ((\exists\alpha)A \strictif (\exists\alpha)B)$ 13, subst

XV. If $\vdash A \strictif B$ then $\vdash (\exists\alpha)A \strictif (\exists\alpha)B$.
Like XIII, using 25 in place of 19.

26. $\vdash (\alpha)(AB) \strictif ((\alpha)A\ (\alpha)B)$.
$(AB) \strictif A$ 2
$(\alpha)(AB) \strictif (\alpha)A$ XIII
$(AB) \strictif B$ *12.17*
$(\alpha)(AB) \strictif (\alpha)B$ XIII
$(\alpha)(AB) \strictif ((\alpha)A\ (\alpha)B)$ XII

27. $\vdash (\exists\alpha)(AB) \strictif ((\exists\alpha)A\ (\exists\alpha)B)$.
Like 26 using XV in place of XIII.

28. $\vdash ((\alpha)A\ (\alpha)B) \strictif (\alpha)(AB)$.
$(\alpha)A \strictif A$ 8
$(\alpha)B \strictif B$ 8
$((\alpha)A\ (\alpha)B) \strictif (AB)$ IX
$(\alpha)(((\alpha)A\ (\alpha)B) \strictif (AB)) \strictif (((\alpha)A\ (\alpha)B) \strictif (\alpha)(AB))$ 22
$(\alpha)(((\alpha)A\ (\alpha)B) \strictif (AB))$ gen
$((\alpha)A\ (\alpha)B) \strictif (\alpha)(AB)$ mod pon

29. $\vdash ((\alpha)A\ (\alpha)B) \equiv (\alpha)(AB)$.
$(((\alpha)A\ (\alpha)B) \strictif (\alpha)(AB))((\alpha)(AB) \strictif ((\alpha)A\ (\alpha)B))$ 28, 26, adj
$((\alpha)A\ (\alpha)B) \equiv (\alpha)(AB)$ def

30. $\vdash (\exists\alpha)A \strictif (\exists\alpha)(A \vee B)$.
$A \strictif (A \vee B)$ *13.2*
$(\exists\alpha)A \strictif (\exists\alpha)(A \vee B)$ XV

31. $\vdash (\exists\alpha)B \prec (\exists\alpha)(A \vee B)$.
Like 30 using *13.21* in place of *13.2*.

32. $\vdash ((\exists\alpha)A \vee (\exists\alpha)B) \prec (\exists\alpha)(A \vee B)$.
$(((\exists\alpha)A \prec (\exists\alpha)(A \vee B))((\exists\alpha)B \prec (\exists\alpha)(A \vee B))) \prec$
$(((\exists\alpha)A \vee (\exists\alpha)B) \prec (\exists\alpha)(A \vee B))$ *19.65*
$((\exists\alpha)A \prec (\exists\alpha)(A \vee B))((\exists\alpha)B \prec (\exists\alpha)(A \vee B))$ 30, 31, adj
$((\exists\alpha)A \vee (\exists\alpha)B) \prec (\exists\alpha)(A \vee B)$ mod pon

33. $\vdash (\exists\alpha)(A \vee B) \equiv ((\exists\alpha)A \vee (\exists\alpha)B)$.
$((\alpha)\sim A\ (\alpha)\sim B) \equiv (\alpha)(\sim A \sim B)$ 29
$\sim(\sim(\alpha)\sim A \vee \sim(\alpha)\sim B) \equiv (\alpha)\sim(\sim\sim A \vee \sim\sim B)$
14.21, subst, def
$\sim\sim(\sim(\alpha)\sim A \vee \sim(\alpha)\sim B) \equiv \sim(\alpha)\sim(\sim\sim A \vee \sim\sim B)$ XI
$(\sim(\alpha)\sim A \vee \sim(\alpha)\sim B) \equiv \sim(\alpha)\sim(A \vee B)$ *12.3*, subst
$((\exists\alpha)A \vee (\exists\alpha)B) \equiv (\exists\alpha)(A \vee B)$ 13, subst

34. $\vdash (\alpha)A \prec (\alpha)(A \vee B)$.
Like 30 using XIII instead of XV.

35. $\vdash (\alpha)B \prec (\alpha)(A \vee B)$.
Like 31 using XIII in place of XV.

36. $\vdash ((\alpha)A \vee (\alpha)B) \prec (\alpha)(A \vee B)$.
Like 32 using 34 and 35 in place of 30 and 31.

Extension of the rule of substitution.

XVI. If the wff's Γ, E, and B are such that B results from A by the substitution of E for one or more occurrences of Γ in A and if $(\alpha_1)(\alpha_2) \cdots (\alpha_n)(\Gamma \equiv E)$ where $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a complete list of the free variables in Γ and E then we may infer B from A and A from B.
$(\alpha_1)(\alpha_2) \cdots (\alpha_n)(\Gamma \equiv E) \prec (\Gamma \equiv E)$
successive applications of 8
$(\alpha_1)(\alpha_2) \cdots (\alpha_n)(\Gamma \equiv E)$ hyp
$\Gamma \equiv E$ mod pon
XVI follows from the rule of substitution.

XVII. If $\vdash A \prec B$ and α does not occur free in A then $\vdash A \prec (\alpha)B$.
$A \prec B$ hyp
$(\alpha)(A \prec B)$ gen
$(\alpha)(A \prec B) \prec (A \prec (\alpha)B)$, where as above. 22
$A \prec (\alpha)B$ mod pon

XVIII. If $\vdash A \prec B$ and α does not occur free in B then $\vdash (\exists\alpha)A \prec B$.
Proof like XVII, using 23 instead of 22.

37. $\vdash (\exists\alpha)\Diamond A \prec \Diamond(\exists\alpha)A$.
$A \prec (\exists\alpha)A$ 16
$\Diamond A \prec \Diamond(\exists\alpha)A$ VI
$(\exists\alpha)\Diamond A \prec \Diamond(\exists\alpha)A$ XVIII

38. ⊦ (∃α)◇A ≡ ◇(∃α)A.

((∃α)◇A ⥽ ◇(∃α)A)(◇(∃α)A ⥽ (∃α)◇A) 37, 11, adj
(∃α)◇A ≡ ◇(∃α)A def

39. ⊦ (α)∼◇∼A ≡ ∼◇∼(α)A.

(∃α)◇∼A ≡ ◇(∃α)∼A 38
∼(α)∼◇∼A ≡ ◇∼(α)∼∼A 13, subst
∼∼(α)∼◇∼A ≡ ∼◇∼(α)∼∼A XI
(α)∼◇∼A ≡ ∼◇∼(α)A *12.3*, subst

40. ⊦ ◇(α)A ⥽ (α)◇A.

(α)A ⥽ A 8
◇(α)A ⥽ ◇A VI
◇(α)A ⥽ (α)◇A XVII

41. ⊦ (α)A ⥽ (β)B, where no free occurrence of α in A is in a well-formed part of A of the form (β)Γ, B is formed from A by replacing all free occurrences of α in A by β, and there is no free occurrence of β in A.

(α)A ⥽ B 8
(α)A ⥽ (β)B XVII

42. ⊦ (α)A ≡ (β)B, where as in 41.

(α)A ⥽ (β)B 41
(β)B ⥽ (α)A 41
(α)A ≡ (β)B adj and def

43. ⊦ (∃α)A ≡ (∃β)B, where as in 41.

(α)∼A ≡ (β)∼B 42
∼(α)∼A ≡ ∼(β)∼B XI
(∃α)A ≡ (∃β)B 13, subst

44. ⊦ (α)(A ≡ B) ⥽ ((∃α)A ≡ (∃α)B).

(α)(A ⥽ B) ⥽ ((∃α)A ⥽ (∃α)B) 25
(α)(B ⥽ A) ⥽ ((∃α)B ⥽ (∃α)A) 25
((α)(A ⥽ B) (α)(B ⥽ A)) ⥽ (((∃α)A ⥽ (∃α)B)((∃α)B ⥽ (∃α)A)) IX
(α)((A ⥽ B)(B ⥽ A)) ⥽ (((∃α)A ⥽ (∃α)B)((∃α)B ⥽ (∃α)A)) 29, subst
(α)(A ≡ B) ⥽ ((∃α)A ≡ (∃α)B) def

45. ⊦ (A ⥽ (α)B) ⥽ (α)(A ⥽ B), where α is not free in A.

(∃α)(A ∼B) ⥽ ((∃α)A (∃α)∼B) 27
(∃α)(A ∼B) ⥽ (A (∃α)∼B) 21, subst
(∃α)(A ∼B) ⥽ (A ∼(α)∼∼B) 13, subst
(∃α)(A ∼B) ⥽ (A ∼(α)B) *12.3*, subst
∼(A∼(α)B) ⥽ ∼(∃α)(A∼B) X
(A ⊃ (α)B) ⥽ ∼(∃α)∼(A ⊃ B) *14.12*, subst
∼◇∼(A ⊃ (α)B) ⥽ ∼◇∼(α)(A ⊃ B) 13, subst, VII
∼◇∼(A ⊃ (α)B) ⥽ (α)∼◇∼(A ⊃ B) 39, subst
(A ⥽ (α)B) ⥽ (α)(A ⥽ B) *18.7*, subst

46. $\vdash (A \prec (\alpha)B) \equiv (\alpha)(A \prec B)$, where α is not free in A.

$((A \prec (\alpha)B) \prec (\alpha)(A \prec B))((\alpha)(A \prec B) \prec (A \prec (\alpha)B))$, where as above. 22, 45, adj

$(A \prec (\alpha)B) \equiv (\alpha)(A \prec B)$ def

47. $\vdash (\exists\alpha)(B \prec A) \prec ((\alpha)B \prec A)$, where α is not free in A.

$(\alpha)(B \sim A) \prec (B \sim A)$ 8

$((\alpha)B\ (\alpha){\sim}A) \prec (B \sim A)$ 29, subst

$((\alpha)B \sim A) \prec (B \sim A)$ 20, subst

$\Diamond((\alpha)B \sim A) \prec \Diamond(B \sim A)$ VI

$\sim\Diamond(B \sim A) \prec \sim\Diamond((\alpha)B \sim A)$ X

$(\exists\alpha)\sim\Diamond(B \sim A) \prec \sim\Diamond((\alpha)B \sim A)$ XVIII

$(\exists\alpha)(B \prec A) \prec ((\alpha)B \prec A)$ def

48. $\vdash (\exists\alpha)(A \prec B) \prec (A \prec (\exists\alpha)B)$, where α is not free in A.

$B \prec (\exists\alpha)B$ 16

$\sim(\exists\alpha)B \prec \sim B$ X

$(\sim(\exists\alpha)B \prec \sim B) \prec ((A \sim(\exists\alpha)B) \prec (A \sim B))$ *19.6*, *12.15*, subst

$(A \sim(\exists\alpha)B) \prec (A \sim B)$ mod pon

$\Diamond(A \sim(\exists\alpha)B) \prec \Diamond(A \sim B)$ VI

$\sim\Diamond(A \sim B) \prec \sim\Diamond(A \sim(\exists\alpha)B)$ X

$(\exists\alpha)\sim\Diamond(A \sim B) \prec \sim\Diamond(A \sim(\exists\alpha)B)$ XVIII

$(\exists\alpha)(A \prec B) \prec (A \prec (\exists\alpha)B)$ def

The above two theorems can be regarded as special cases of the more general principle $(\exists\alpha)\sim\Diamond\sim A \prec \sim\Diamond\sim(\exists\alpha)A$.[6] Their converses are not provable.

49. $\vdash (\exists\alpha)(A \prec B) \prec ((\alpha)A \prec (\exists\alpha)B)$.

$\sim(\exists\alpha)B \prec \sim B$ step 2 of 48

$(\alpha)A \prec A$ 8

$((\alpha)A \sim(\exists\alpha)B) \prec (A \sim B)$ IX

$\Diamond((\alpha)A \sim(\exists\alpha)B) \prec \Diamond(A \sim B)$ VI

$\sim\Diamond(A \sim B) \prec \sim\Diamond((\alpha)A \sim(\exists\alpha)B)$ XI

$(\exists\alpha)\sim\Diamond(A \sim B) \prec \sim\Diamond((\alpha)A \sim(\exists\alpha)B)$ XVIII

$(\exists\alpha)(A \prec B) \prec ((\alpha)A \prec (\exists\alpha)B)$ def

50. $\vdash (\alpha)(B \prec A) \prec ((\alpha)B \prec A$.

$(\alpha)B \prec B$ 8

$((\alpha)B \prec B) \prec (((\alpha)B \sim A) \prec (B \sim A))$ *19.6*

$((\alpha)B \sim A) \prec (B \sim A)$ mod pon

$\Diamond((\alpha)B \sim A) \prec \Diamond(B \sim A)$ VI

$\sim\Diamond(B \sim A) \prec \sim\Diamond((\alpha)B \sim A)$ XI

$(\alpha)\sim\Diamond(B \sim A) \prec (\alpha)\sim\Diamond((\alpha)B \sim A)$ XIII

$(\alpha)\sim\Diamond((\alpha)B \sim A) \prec \sim\Diamond((\alpha)B \sim A)$ 8

$(\alpha)\sim\Diamond(B \sim A) \prec \sim\Diamond((\alpha)B \sim A)$ VIII

$(\alpha)(B \prec A) \prec ((\alpha)B \prec A)$ def

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[6] $(\exists\alpha)\sim\Diamond\sim A \prec \sim\Diamond\sim(\exists\alpha)A$ follows easily from 40.

51. $\vdash (\alpha)(A \vee B) \equiv (A \vee (\alpha)B)$, where α is not free in A.

$((\alpha)A \vee (\alpha)B) \strictif (\alpha)(A \vee B)$ 36

$(A \vee (\alpha)B) \strictif (\alpha)(A \vee B)$, where etc. 20, subst

Lemma:[7] $\vdash (\alpha)(\Gamma \supset E) \strictif (\Gamma \supset (\alpha)E)$ where α is not free in Γ.

$(\alpha)(\Gamma \supset E) \strictif ((\alpha)\Gamma \supset (\alpha)E)$ 9

$(\alpha)(\Gamma \supset E) \strictif (\Gamma \supset (\alpha)E)$ 20, subst

$(\alpha)(\sim A \supset B) \strictif (\sim A \supset (\alpha)B)$ Lemma

$(\alpha)(A \vee B) \strictif (A \vee (\alpha)B)$ *14.2*, subst

$((\alpha)(A \vee B) \strictif (A \vee (\alpha)B))((A \vee (\alpha)B) \strictif (\alpha)(A \vee B))$ adj

$(\alpha)(A \vee B) \equiv (A \vee (\alpha)B)$ def

52. $\vdash (\alpha)(A \vee B) \strictif ((\alpha)A \vee (\exists\alpha)B)$.

Lemma: $\vdash (\alpha)(A \supset B) \strictif ((\exists\alpha)A \supset (\exists\alpha)B)$.

Like 25 using 9 instead of 19.

$(\alpha)(\sim A \supset B) \strictif ((\exists\alpha)\sim A \supset (\exists\alpha)B)$ Lemma

$(\alpha)(A \vee B) \strictif (\sim(\exists\alpha)\sim A \vee (\exists\alpha)B)$ *14.12*, *12.3*, subst

$(\alpha)(A \vee B) \strictif ((\alpha)A \vee (\exists\alpha)B)$ def

53. $\vdash (A\,(\exists\alpha)B) \equiv (\exists\alpha)(AB)$, where α is not free in A.

$(\alpha)(\sim A \vee \sim B) \equiv (\sim A \vee (\alpha)\sim B)$, where etc. 51

$(\alpha)\sim(\sim\sim A\ \sim\sim B) \equiv \sim(\sim\sim A\ \sim(\alpha)\sim B)$ def

$(\alpha)\sim(AB) \equiv \sim(A\ \sim(\alpha)\sim B)$ 12.3, subst

$\sim(\alpha)\sim(AB) \equiv \sim\sim(A\ \sim(\alpha)\sim B)$ XI

$(\exists\alpha)(AB) \equiv (A\,(\exists\alpha)B)$ *12.3*, subst, 13

54. $\vdash (\alpha)\sim\Diamond A \strictif (\alpha)(A \strictif B)$.

$\sim\Diamond A \strictif (A \strictif B)$ *19.74*

$(\alpha)\sim\Diamond A \strictif (\alpha)(A \strictif B)$ XIII

55. $\vdash \sim\Diamond(\exists\alpha)A \strictif (\alpha)(A \strictif B)$.

$(\alpha)\sim\Diamond A \strictif (\alpha)(A \strictif B)$ 54

$\sim(\exists\alpha)\sim\sim\Diamond A \strictif (\alpha)(A \strictif B)$ def

$\sim(\exists\alpha)\Diamond A \strictif (\alpha)(A \strictif B)$ *12.3*, subst

$\sim\Diamond(\exists\alpha)A \strictif (\alpha)(A \strictif B)$ 38, subst

56. $\vdash \sim\Diamond\sim(\alpha)A \strictif (\alpha)(B \strictif A)$.

$\sim\Diamond\sim A \strictif (B \strictif A)$ *19.75*

$(\alpha)\sim\Diamond\sim A \strictif (\alpha)(B \strictif A)$ XIII

$\sim\Diamond\sim(\alpha)A \strictif (\alpha)(B \strictif A)$ 39, subst

57. $\vdash \sim\Diamond(\exists\alpha)(AB) \strictif (\alpha)(A \strictif \sim B)$.

$\sim\Diamond(\exists\alpha)(AB) \strictif \sim\Diamond(\exists\alpha)(AB)$ *12.1*

$\sim\Diamond(\exists\alpha)(AB) \strictif \sim(\exists\alpha)\Diamond(AB)$ 38, subst

$\sim\Diamond(\exists\alpha)(AB) \strictif \sim(\exists\alpha)\sim(A \strictif \sim B)$ def, *12.3*, subst

$\sim\Diamond(\exists\alpha)(AB) \strictif (\alpha)(A \strictif \sim B)$ def

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[7] This Lemma might have been used as an axiom schema in place of 9 and 10.

58. $\vdash (\alpha)(A \prec (B \vee \Gamma)) \equiv \sim\Diamond\sim(B \vee (\alpha)(A \supset \Gamma))$, where α is not free in B.

$(\alpha)\sim\Diamond(A \sim(B \vee \Gamma)) \equiv (\alpha)\sim\Diamond(A \sim(B \vee \Gamma))$ *12.11*

$(\alpha)\sim\Diamond(A \sim(B \vee \Gamma)) \equiv (\alpha)\sim\Diamond\sim(\sim A \vee \sim\sim(B \vee \Gamma))$ *14.21*, subst

$(\alpha)\sim\Diamond(A \sim(B \vee \Gamma)) \equiv (\alpha)\sim\Diamond\sim(\sim A \vee (B \vee \Gamma))$ *12.3*, subst

$(\alpha)\sim\Diamond(A \sim(B \vee \Gamma)) \equiv (\alpha)\sim\Diamond\sim(B \vee (\sim A \vee \Gamma))$ *13.41*, subst

$(\alpha)\sim\Diamond(A \sim(B \vee \Gamma)) \equiv \sim\Diamond\sim(\alpha)(B \vee (\sim A \vee \Gamma))$ 39, subst

$(\alpha)\sim\Diamond(A \sim(B \vee \Gamma)) \equiv \sim\Diamond\sim(B \vee (\alpha)(\sim A \vee \Gamma))$ 51, subst

$(\alpha)(A \prec (B \vee \Gamma)) \equiv \sim\Diamond\sim(B \vee (\alpha)(A \supset \Gamma))$ def, *14.2*, subst

59. $\vdash ((\alpha)(A \prec B)\ (\alpha)(B \prec \Gamma)) \prec (\alpha)(A \prec \Gamma)$.

$((A \prec B)(B \prec \Gamma)) \prec (A \prec \Gamma)$ 5

$(\alpha)((A \prec B)(B \prec \Gamma)) \prec (\alpha)(A \prec \Gamma)$ XIII

$((\alpha)(A \prec B)\ (\alpha)(B \prec \Gamma)) \prec (\alpha)(A \prec \Gamma))$ 29, subst

60. $\vdash ((\alpha)(A \prec B)\ \Gamma) \prec E$, where Γ is the result of replacing all free occurrences of α in A by β, and α does not occur in a well-formed part of $A \prec B$ of the form $(\beta)H$, and E is the result of replacing all free occurrences of α in B by β.

Lemma: If $\vdash A \prec (B \prec \Gamma)$ then $\vdash (AB) \prec \Gamma$.

$(A \prec (B \prec \Gamma) \prec ((BA) \prec (B(B \prec \Gamma)))$ *19.6*, *12.15*, subst

$A \prec (B \prec \Gamma)$ hyp

$(BA) \prec (B(B \prec \Gamma))$ mod pon

$(B(B \prec \Gamma)) \prec \Gamma$ 6

$(AB) \prec \Gamma$ VIII, *12.15*, subst

$(\alpha)(A \prec B) \prec (\Gamma \prec E)$, where as above. 8

$((\alpha)(A \prec B)\Gamma) \prec E$ Lemma

59 and 60 are two instances of the syllogism in Barbara.

61. $\vdash ((\alpha)(A \prec B)\ (\alpha)(A \prec \Gamma)) \equiv (\alpha)(A \prec (B\Gamma))$.

$((A \prec B)(A \prec \Gamma)) \prec (A \prec (B\Gamma))$ *19.63*

$(\alpha)((A \prec B)(A \prec \Gamma)) \prec (\alpha)(A \prec (B\Gamma))$ XIV

$((\alpha)(A \prec B)\ (\alpha)(A \prec \Gamma)) \prec (\alpha)(A \prec (B\Gamma))$ 29, subst

62. $\vdash ((\alpha)(A \prec B)\ (\exists\alpha)A) \prec (\exists\alpha)B$.

$(\alpha)(A \prec B) \prec ((\exists\alpha)A \prec (\exists\alpha)B)$ 25

$((\alpha)(A \prec B)\ (\exists\alpha)A) \prec (\exists\alpha)B$ Lemma of 60

63. $\vdash ((\alpha)(A \equiv B)\ (\alpha)(B \equiv \Gamma)) \prec (\alpha)(A \equiv \Gamma)$.

$((A \equiv B)(B \equiv \Gamma)) \prec (A \equiv \Gamma)$ 5, IX, def

Remainder of proof like 59.

64. $\vdash (\alpha)(A \equiv B) \equiv (\alpha)(\sim B \equiv \sim A)$.

Steps 1, 2 of XI, XIV, and def.

65. $\vdash ((\alpha)(\mathrm{A} \strictif \mathrm{B}) \sim\Diamond\sim(\alpha)\mathrm{A}) \strictif \sim\Diamond\sim(\alpha)\mathrm{B}.$

$((\mathrm{A} \strictif \mathrm{B}) \sim\Diamond\sim\mathrm{A}) \strictif \sim\Diamond\sim\mathrm{B}$ *18.53*

$(\alpha)((\mathrm{A} \strictif \mathrm{B}) \sim\Diamond\sim\mathrm{A}) \strictif (\alpha)\sim\Diamond\sim\mathrm{B}$ XIII

$((\alpha)(\mathrm{A} \strictif \mathrm{B}) (\alpha)\sim\Diamond\sim\mathrm{A}) \strictif (\alpha)\sim\Diamond\sim\mathrm{B}$ 29, subst

$((\alpha)(\mathrm{A} \strictif \mathrm{B}) \sim\Diamond\sim(\alpha)\mathrm{A}) \strictif \sim\Diamond\sim(\alpha)\mathrm{B}$ 39, subst

66. $\vdash ((\alpha)(\mathrm{A} \strictif \mathrm{B}) (\alpha)\Diamond\mathrm{A}) \strictif (\alpha)\Diamond\mathrm{B}.$

Like 65 using *18.51* instead of *18.53*.

67. $\vdash (\sim\Diamond\sim(\alpha)\mathrm{A} (\alpha)((\mathrm{AB}) \strictif \Gamma)) \strictif (\alpha)(\mathrm{B} \strictif \Gamma).$

$(\sim\Diamond\sim\mathrm{A} ((\mathrm{AB}) \strictif \Gamma)) \strictif (\mathrm{B} \strictif \Gamma)$ *18.61*

$(\alpha)(\sim\Diamond\sim\mathrm{A} ((\mathrm{AB}) \strictif \Gamma)) \strictif (\alpha)(\mathrm{B} \strictif \Gamma)$ XIII

$((\alpha)\sim\Diamond\sim\mathrm{A} (\alpha)((\mathrm{AB}) \strictif \Gamma)) \strictif (\alpha)(\mathrm{B} \strictif \Gamma)$ 29, subst

$(\sim\Diamond\sim(\alpha)\mathrm{A} (\alpha)((\mathrm{AB}) \strictif \Gamma)) \strictif (\alpha)(\mathrm{B} \strictif \Gamma)$ 39, subst

68. $\vdash ((\exists\alpha)\Diamond\mathrm{A} \vee (\exists\alpha)\Diamond\mathrm{B}) \equiv (\exists\alpha)\Diamond(\mathrm{A} \vee \mathrm{B}).$

$(\Diamond\mathrm{A} \vee \Diamond\mathrm{B}) \equiv \Diamond(\mathrm{A} \vee \mathrm{B})$ *19.82*

$(\exists\alpha)(\Diamond\mathrm{A} \vee \Diamond\mathrm{B}) \equiv (\exists\alpha)\Diamond(\mathrm{A} \vee \mathrm{B})$ Like proof of XIV.

$((\exists\alpha)\Diamond\mathrm{A} \vee (\exists\alpha)\Diamond\mathrm{B}) \equiv (\exists\alpha)\Diamond(\mathrm{A} \vee \mathrm{B})$ 33, subst

69. $\vdash (\Diamond(\exists\alpha)\mathrm{A} \vee \Diamond(\exists\alpha)\mathrm{B}) \equiv \Diamond(\exists\alpha)(\mathrm{A} \vee \mathrm{B}).$ 68, subst, 38

70. $\vdash (\sim\Diamond\sim(\alpha)\mathrm{A} \sim\Diamond\sim(\alpha)\mathrm{B}) \strictif (\alpha)(\mathrm{A} \equiv \mathrm{B}).$

Like 65 using *19.84* instead of *18.53*.

71. $\vdash (\sim\Diamond(\exists\alpha)\mathrm{A} \sim\Diamond(\exists\alpha)\mathrm{B}) \strictif (\alpha)(\mathrm{A} \equiv \mathrm{B}).$

$(\sim\Diamond\mathrm{A} \sim\Diamond\mathrm{B}) \strictif (\mathrm{A} \equiv \mathrm{B})$ *19.83*

$(\alpha)(\sim\Diamond\mathrm{A} \sim\Diamond\mathrm{B}) \strictif (\alpha)(\mathrm{A} \equiv \mathrm{B})$ XIII

$((\alpha)\sim\Diamond\mathrm{A} (\alpha)\sim\Diamond\mathrm{B}) \strictif (\alpha)(\mathrm{A} \equiv \mathrm{B})$ 29, subst

$(\sim(\exists\alpha)\sim\sim\Diamond\mathrm{A} \sim(\exists\alpha)\sim\sim\Diamond\mathrm{B}) \strictif (\alpha)(\mathrm{A} \equiv \mathrm{B})$ def

$(\sim(\exists\alpha)\Diamond\mathrm{A} \sim(\exists\alpha)\Diamond\mathrm{B}) \strictif (\alpha)(\mathrm{A} \equiv \mathrm{B})$ *12.3*, subst

$(\sim\Diamond(\exists\alpha)\mathrm{A} \sim\Diamond(\exists\alpha)\mathrm{B}) \strictif (\alpha)(\mathrm{A} \equiv \mathrm{B})$ 38, subst

Definition: $\Box\mathrm{A} =_{\mathrm{df}} \sim\Diamond\sim\mathrm{A}.$

72. $\vdash \Box(\alpha_1)(\alpha_2) \cdots (\alpha_m)\mathrm{A} \equiv (\alpha_1)(\alpha_2) \cdots (\alpha_m)\Box\mathrm{A}.$

39, successive applications of subst.

The following theorems are not proved in *Symbolic logic*. Some of them will be useful in proving XIX below.

73. $\vdash (\mathrm{A} \strictif \mathrm{B}) \strictif (\Diamond\mathrm{A} \supset \Diamond\mathrm{B}).$

$(((\mathrm{A} \strictif \mathrm{B}) \Diamond\mathrm{A}) \strictif \Diamond\mathrm{B}) \strictif (((\mathrm{A} \strictif \mathrm{B}) \Diamond\mathrm{A}) \strictif \Diamond\mathrm{B})$ *12.1*

$(((\mathrm{A} \strictif \mathrm{B}) \Diamond\mathrm{A}) \strictif \Diamond\mathrm{B}) \strictif \sim\Diamond(((\mathrm{A} \strictif \mathrm{B}) \Diamond\mathrm{A}) \sim\Diamond\mathrm{B}))$ def

$(((\mathrm{A} \strictif \mathrm{B}) \Diamond\mathrm{A}) \strictif \Diamond\mathrm{B}) \strictif \sim\Diamond((\mathrm{A} \strictif \mathrm{B})(\Diamond\mathrm{A} \sim\Diamond\mathrm{B}))$ *12.5*, subst

$(((\mathrm{A} \strictif \mathrm{B}) \Diamond\mathrm{A}) \strictif \Diamond\mathrm{B}) \strictif ((\mathrm{A} \strictif \mathrm{B}) \strictif \sim(\Diamond\mathrm{A} \sim\Diamond\mathrm{B}))$ def, *12.3*, subst

$(((\mathrm{A} \strictif \mathrm{B}) \Diamond\mathrm{A}) \strictif \Diamond\mathrm{B}) \strictif ((\mathrm{A} \strictif \mathrm{B}) \strictif (\Diamond\mathrm{A} \supset \Diamond\mathrm{B}))$ def

$((\mathrm{A} \strictif \mathrm{B}) \Diamond\mathrm{A}) \strictif \Diamond\mathrm{B}$ *18.51*

$(\mathrm{A} \strictif \mathrm{B}) \strictif (\Diamond\mathrm{A} \supset \Diamond\mathrm{B})$ mod pon

74. $\vdash (A \prec B) \prec (\Box A \supset \Box B)$.
Same method as 73 using *18.53* in place of *18.51*.

75. $\vdash (A \prec B) \prec (\sim\Diamond B \supset \sim\Diamond A)$.
Same method as 73 using *18.5*.

76. $\vdash (A \equiv B) \prec (\Diamond A \equiv \Diamond B)$.
$((A \prec B) \prec (\Diamond A \supset \Diamond B))((B \prec A) \prec (\Diamond B \supset \Diamond A))$ 73, adj
$((A \prec B)(B \prec A)) \prec ((\Diamond A \supset \Diamond B)(\Diamond B \supset \Diamond A)$ *19.68*, mod pon
$(A \equiv B) \prec (\Diamond A \equiv \Diamond B)$ def

77. $\vdash (A \equiv B) \prec (\Box A \equiv \Box B)$.
Like 76, using 74.

78. $\vdash (A \equiv B) \prec (\sim\Diamond A \equiv \sim\Diamond B)$.
Like 76 using 75.

79. $\vdash (A \equiv B) \equiv (\sim A \equiv \sim B)$. *12.1*, *12.44*, subst, *12.15*, def

80. $\vdash ((A \equiv B)(\Gamma \equiv H)) \prec ((A\Gamma) \equiv (BH))$.
19.68, adj, IX, *12.5*, *12.15*, subst

81. $\vdash (A \equiv B) \prec ((A\Gamma) \equiv (B\Gamma))$. *19.6*, adj, *19.68*, mod pon, def

82. $\vdash \Box (A \equiv B) \equiv (A \equiv B)$.
$\Box(A \supset B) \equiv (A \prec B)$ *18.7*
$\Box(B \supset A) \equiv (B \prec A)$ *18.7*
$(\Box(A \supset B)\ \Box(B \supset A)) \prec ((A \prec B)(B \prec A))$ IX
$\Box(A \equiv B) \equiv (A \equiv B)$ *19.81*, subst, def

Let every wff occurring in A, as well as A itself, be said to be within the scope of the left-hand diamond in the expression $\Diamond A$. An "nth degree occurrence of Γ in A" will be an occurrence of Γ which falls within the scope of not more than n diamonds in A. Abbreviate A as $\Box^0 A$ and $\Box\Box^{n-1} A$ as $\Box^n A$.

XIX. If the wff's Γ, E, B are such that B results from A by the substitution of E for zero or more nth degree occurrences of Γ in A and if $\alpha_1, \alpha_2, \cdots, \alpha_m$ is a complete list of the free variables in Γ and E, then

$$\vdash \Box^n (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A \equiv B).$$

Proof by induction on n. When n is zero XIX becomes

$$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A \equiv B), \text{ where as above,}$$

which we will call XIX_0.

XIX_0 is provable by an induction on the number of occurrences of the primitive symbols $\sim$, $\cdot$, $(\exists\ )$, $\Diamond$ in A.

Suppose the number of occurrences of these symbols in A is zero. Then, XIX_0 is one of the following:

(1). $\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (\Gamma \equiv E)$

(2). $\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A \equiv A)$.

Case (1) is provable by successive applications of 8 and VIII.

Case (2). $\vdash \sim\Diamond(A \sim A) \prec ((A \sim A) \prec \sim(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E))$ *19.74*

$\vdash \sim\Diamond(A \sim A)$ *18.8*

$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec \sim(A \sim A)$ mod pon, X, *12.3*, subst

$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A \supset A)$ def

$\vdash \Box(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec \Box(A \equiv A)$ VII, *12.7*, def, subst

$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A \equiv A)$ 72, 82, subst

Assume that XIX_0 is true when A contains not more than i occurrences of $\sim$, , ($\exists$), and $\Diamond$. If A contains $i + 1$ of these symbols it must be $\sim A_1$, $A_1 \cdot A_2$, $(\exists\alpha)A_1$ or $\Diamond A_1$. XIX_0 becomes correspondingly

(1.1). $\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (\sim A_1 \equiv \sim B_1)$,

(2.1). $\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec ((A_1 \cdot A_2) \equiv (B_1 \cdot B_2))$,

(3.1). $\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec ((\exists\alpha)A_1 \equiv (\exists\alpha)B_1)$,

(4.1). $\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (\Diamond A_1 \equiv \Diamond B_1)$,

where B_1 and B_2 result from the substitution of E for zero or more zero degree occurrences of Γ in A_1 and A_2 respectively.

The following proof assumes that Γ is not identical with A. If it is, the proof is the same as Case (1) above. Where Γ does not occur in A the proof is like that of Case (2).

Case (1.1) follows immediately from the hypothesis, 79, and subst.

Case (2.1). $\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A_1 \equiv B_1)$ hyp

$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A_2 \equiv B_2)$ hyp

$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec ((A_1 \equiv B_1)\ (A_2 \equiv B_2))$ XII

$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec ((A_1 \cdot A_2) \equiv (B_1 \cdot B_2)$ 80, VIII

Case (3.1). $\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A_1 \equiv B_1)$ hyp

$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (\alpha)(A_1 \equiv B_1)$ hyp, XIII

$(\alpha)(A_1 \equiv B_1) \prec ((\exists\alpha)A_1 \equiv (\exists\alpha)B_1)$ 44

$(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec ((\exists\alpha)A_1 \equiv (\exists\alpha)B_1)$ VIII

Case (4.1). On the hypothesis n is zero, no occurrence of Γ in A is replaced by E. This case then becomes

$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (\Diamond A_1 \equiv \Diamond A_1)$

Proof like Case (2).

Suppose XIX is provable for $n = k$. We shall then prove it for $n = k + 1$. We will assume that Γ is not identical with A and that it occurs in A. If Γ is identical with A the proof is like Case 1) using in addition, successive applications of *18.42*. If Γ does not occur in A the proof is like Case (2), using $\Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$ in place of $(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$.

First it is necessary to prove the case where A is $\Diamond A_1$, B is $\Diamond B_1$, and where

A_1 results from B_1 by replacing by E zero or more kth degree occurrences of Γ in A. The other cases follow from an induction on the number of "additional occurrences" in A. An "additional occurrence" in A is an occurrence (of a primitive symbol) which is not part of a wff in A of the form $\Diamond H$.

Case (1.2). $\vdash \Box^k(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A_1 \equiv B_1)$ hyp

$\vdash (A_1 \equiv B_1) \prec (\Diamond A_1 \equiv \Diamond B_1)$ 76

$\vdash \Box^k(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (\Diamond A_1 \equiv \Diamond B_1)$ VIII

$\vdash \Box\Box^k(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec \Box(\Diamond A_1) \equiv \Diamond B_1)$ VII

$\vdash \Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (\Diamond A_1 \equiv \Diamond B_1)$
def, 72, 82, subst

Case (1.2) is the proof of XIX for zero additional occurrences of the primitive symbols in A. Suppose XIX is true for not more than i additional occurrences of the primitive symbols in A. We will now show that it is true for $i + 1$ additional occurrences.

If A contains $i + 1$ additional occurrences it must be $\sim A_1$, $A_1 \cdot A_2$, or $(\exists\alpha)A_1$. XIX becomes correspondingly

(1.3). $\vdash \Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (\sim A_1 \equiv \sim B_1)$,

(2.3). $\vdash \Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec ((A_1 \cdot A_2) \equiv (B_1 \cdot B_2)$,

(3.3). $\vdash \Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec ((\exists\alpha)A_1 \equiv (\exists\alpha)B_1)$,

where B_1 and B_2 result from the substitution of E for zero or more $k + 1$ degree occurrences of Γ in A_1 and A_2 respectively.

The proofs of (1.3), (2.3), and (3.3) are like those of Cases (1.1), (2.1), and (3.1) replacing $(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$ by $\Box^{k+1}(\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E)$.

The addition of the following axiom schema, $\Diamond\Diamond A \prec \Diamond A$, would make it possible to obtain $\Box\Box A \equiv \Box A$ and hence to prove XIX* as follows:

XIX*. If the wff's Γ, E, and B are such that B results from A by the substitution of E for zero or more occurrences of Γ in A and if $\alpha_1, \alpha_2, \cdots, \alpha_m$ is a complete list of the free variables in and E then

$$\vdash (\alpha_1)(\alpha_2) \cdots (\alpha_m)(\Gamma \equiv E) \prec (A \equiv B).$$

The assumption $\Diamond\Diamond A \prec \Diamond A$ is the distinctive feature of S4[8] in contrast to S2, and if it were included in the present system we could prove XIX* and thus dispense with a consideration of the "degree" of occurrence of Γ in A.

It might be interesting to note some of the differences between this system and the ordinary functional calculus of first order.

The converses of 47 and 48,

$((\alpha)B \prec A) \prec (\exists\alpha)(B \prec A)$, where α is not free in A,

$((A \prec (\exists\alpha)B) \prec (\exists\alpha)(A \prec B)$, where α is not free in A,

[8] See Lewis and Langford, p. 501.

are not provable although their analogues hold in a functional calculus based on material implication.

The theorems corresponding to the so-called paradoxes of material implication such as

$$\sim(\exists\alpha)A \prec (\alpha)(A \prec B)$$

are not provable. However we can derive

$$\sim\Diamond(\exists\alpha)A \prec (\alpha)(A \prec B).$$

Theorems like

$$(\alpha)(A \prec (B \vee \Gamma)) \prec ((\alpha)(A \prec B) \vee (\exists\alpha)(A\Gamma))$$

depend on some such principle as

$$(A \prec (B \vee \Gamma)) \prec (A \prec B) \vee (A \prec \Gamma)),$$

which is not available in S2.

We cannot derive

$$(\alpha)(A \prec (B \vee \Gamma)) \equiv (B \vee (\alpha)(A \prec \Gamma), \text{ where } \alpha \text{ is not free in B},$$

although

$$(\alpha)(A \prec (B \vee \Gamma)) \equiv \sim \Diamond\sim(B \vee (\alpha)(A \supset \Gamma)), \text{ where etc.}$$

is a theorem.[9]

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