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Class 42: The Right Logic?
Fisher 153-161

I. Functions and second-order logic

What we have done this semester is sometimes called baby logic, or classical first-order logic.
There are many extensions or alterations of classical logic.
We have glanced at modal logic and three-valued logic.
A small addition extension would include function symbols: f(x).
Consider terms like 'the father of', 'the successor of', 'the sum of', and 'the teacher of'.
Each takes one or more arguments, from their domain, and produces a single output, the range.
One-place functions take one argument, two-place arguments take two arguments.

the father of: f(x)
the successor of: s(x)
the sum of: s(x, y)
the teacher of: $s(x_1...x_n)$

The last function can take as arguments, say, all the students in the class.
Here are some sentences of first-order logic, with functions:

Olaf loves his mother:	Lom(o)
Olaf loves his grandmothers:	Lom(m(o)) • Lom(f(o))
Noone is his/her own mother:	(x)~x=m(x)

Functions are essential for more complex logical reasoning, but they tread on mathematics
We have not really mentioned second-order logic.
Consider the following sentence:

No two things have all properties in common.
(x)(y)[~x=y ⊃ (∃F)(Fx • ~Fy)]

Note that the predicates are now variables, as well.
Similarly, Leibniz's law is naturally rendered in second-order logic.

(x)(y)[x=y ⊃ (F)(Fx ≡ Fy)]

And the identity of indiscernibles is clearer this way:

(x)(y)[(F)(Fx ≡ Fy) ⊃ x=y]

We can even introduce attributes of attributes.
Consider: All useful properties are desirable:

(F)(**U**F ⊃ **D**F)

Or: A man who possesses all virtues is a virtuous man, but there are virtuous man who do not possess all virtues:

$(x)\{[Mx \bullet (F)(\mathbf{V}F \supset Fx)] \supset Vx\} \bullet (\exists x)[Mx \bullet Vx \bullet (\exists F)(\mathbf{V}F \bullet {\sim}Fx)]$

If we quantify over properties of properties, we create third-order logic.
All logics after first-order logic are called higher-order logic

II. Branching quantifiers

Another extension of classical logic involves branching quantifiers.
Consider: Some relative of each villager and some relative of each townsman hate each other.
We might regiment it as:

$(x)\{Vx \supset (\exists y)\{Ryx \bullet (z)[Tz \supset (\exists w)(Rwz \bullet Hyw \bullet Hwy)]\}\}$

This regimentation makes the choice of the townspeople dependent on the choice of the villagers.
Turning it around makes the choice of the villagers depend on the choice of the townspeople.

$(x)\{Tx \supset (\exists y)\{Ryx \bullet (z)[Vz \supset (\exists w)(Rwz \bullet Hyw \bullet Hwy)]\}\}$

But, what we really want to do is choose each independently, and then express the relation.
In order to do so, we have to pull out the quantifiers, according to the rules of passage:

$(x)(\exists y)(z)(\exists w)[(Vx \supset Ryx) \bullet (Tz \supset Rwz) \bullet Hyw \bullet Hwy]$

Then, we branch the quantifiers:

$$\begin{array}{ll} (x)(\exists y) : & \\ & : \ [(Vx \supset Ryx) \bullet (Tz \supset Rwz) \bullet Hyw \bullet Hwy] \\ (z)(\exists w) : & \end{array}$$

There are other sentences which naturally take branching quantifiers.
Consider: Some book by every author is referred to in some essay by every critic.
The problem with branching quantifiers is essentially the problem with functions.
Standard interpretations of branching quantifiers involve functions.

III. Second-order logic and set theory

Quine argues that any extensions of classical logic are no longer logic.
When we interpret first-order logic, we specify a domain for the variables to range over.
We can discuss restricted domains.
If we want to interpret number theory, for example, we restrict our domain to the integers.
If we want to interpret a biological theory, we might restrict our domain to species, say.
But, for our most general reasoning, for logical theory itself, we take an unrestricted domain.
The domain is the universe, everything there is.
To interpret first-order logic, the domain is the universe of objects.
So consider the sentence, ‘there are blue hats’:

$(\exists x)(Bx. \bullet Hx)$

For this sentence to be true, there has to be a thing, which will substitute for the variable 'x', and which has both the property of being a hat and being blue.
Thus Quine's dictum, 'to be is to be the value of a variable'.
Our most basic commitments arise from examining the domain of quantification for our best theory of everything, physics, say.
Now, consider a second-order logic, 'some properties are shared by two people'.

$(\exists F)(\exists x)(\exists y)(Px \bullet Py \bullet \sim x=y \bullet Fx \bullet Fy)$

For this second-order sentence to be true, there have to be two people, and there has to be a property.
That is, the value of the variable 'F' is not an object, but a property of an object.
If we were Plato, we might not mind such properties.
That is, we could take them to be the forms, or eternal ideas.
But, commitments to properties, in addition to the objects which have those properties, is metaphysically contentious.
The first-order sentence about blue hats referred only to an object with properties.
The second-order sentence reifies properties.
Is there really blueness, in addition to blue things?

The least controversial way to understand properties is to take them to be sets of the object which have those properties.
So 'blueness' would refer to the collection of all blue things.
Thus, second-order logic seems at least to commit to the existence of sets.
We might want sets, if we think there are mathematical objects.
But, if so, we can take them to be objects, values of first-order variables, rather than values of second-order variables.
We can just count them as among the objects in the universe, in the domain of quantification, rather than sneaking them in through the interpretations of second-order variables.
Quine calls second-order logic set theory in sheeps clothing.

On the other hand, in favor of second-order logic, it is difficult to see how one could regiment sentences in first-order logic like the ones I translated into second-order logic above.
Similarly, the sentences for which I urged the use of branching quantifiers are only awkwardly written in first-order logic.

IV. The limits of logic

So, we are looking for a line at which to draw the limits of logic.
On one side of the line would be the true logic.
On the other side of the line would be specific domains, like mathematics, or physics, or metaphysics.
One argument in favor of drawing the line right where we drew it in the course, perhaps the most popular argument, relies on the completeness of first-order logic with identity.
Stronger logics, like second-order logic, lose completeness.

When one first hears the term 'completeness', it sounds really sexy.

First-order logic with identity is provably complete.
But all that really means is that every sentence of first-order logic that you can write can be proven, or disproven.
It says nothing about the expressive completeness, whether there are sentences, like Leibniz's law, that can not be written in first-order logic.

Fisher discusses one final argument in favor of classical logic, and against extensions of it.
It is called the change of logic - change of subject argument.
The basic idea of the argument is that in order to disagree with someone, you have to at least agree on what you are disagreeing about.
There has to be some common ground on which you can stand, to argue, or else you are not really disagreeing at all.
Consider two terms, and their definitions:

> Chair_1: desk chairs, dining room chairs, and such, but not recliners or bean bag chairs
> Chair_2: all chair_1 objects, and also recliners and bean bag chairs

Now, consider one person, who uses 'chair' as chair_1 and another person who uses 'chair' as chair_2.
And imagine them both talking about a bean bag chair.
Person_1 affirms 'that's a chair', while person_2 denies that sentence.
Since they are using the same term, it looks like they are disagreeing, but they are not really disagreeing about whether the bean bag chair is a chair.
They are disagreeing about what 'chair' means.
In order to describe the disagreement over 'chair', they have to at least agree on what it means to affirm or deny a sentence.
If we are considering debates over the correct logic, even those kinds of claims are under discussion.
So, debates over the correct logic seem to be more like the disagreement between chair_1 and chair_2.
The disputants don't even agree on the terms they are using.

Imagine we are linguists, and we are headed to a newly-discovered alien planet.
We have to translate a completely new language into English.
We start by assuming that the aliens obey the rules of logic.
Quine points out that if we were to propose a translation of the alien language on which the aliens often made statements that translated into the form of '$p \bullet {\sim}p$', we would revise our translation.
We need logic to serve as a ground for the translation.
We need common ground even to formulate disagreement.
If we disagree about the right logic, then we have merely changed the subject.

There is much more to say about this topic,
My philosophy of language class will look a bit more at this issue, especially the question of radical translation.

Thank you all for the best logic class I have ever taught.