Class 41: Identity Theory, Derivations (§8.7)

I. **The three ID rules**

We saw that there are three rules governing identity (ID).

1. Reflexivity: a=a
2. Symmetry: a=b :: b=a
3. Indiscernibility of Identicals: $\mathscr{F}$a
   a=b / $\mathscr{F}$b

Reflexivity is an axiom schema.
Symmetry and indiscernibility are rules of replacement.
Thus, we use them differently.
We can add an instance of the axiom schema into any proof, with no line justification.
We can use symmetry on whole lines or on parts of lines.
With indiscernibility, we are always re-writing a whole line, switching one constant for another.

II. **Derivations in identity theory**

Consider the original problem:

Superman can fly.
Superman is Clark Kent.
∴ Clark Kent can fly.

1. Fs	
2. s=c	/ Fc
3. Fc	1, 2, ID

QED

Using the symmetry rule:

1. a=b ⊃ j=k	
2. b=a	
3. Fj	/ Fk
4. a=b	2, Id
5. j=k	1, 4, MP
6. Fk	3, 5, Id

QED

To derive the negation of an identity statement, one often uses IP:

1. Rm
2. ~Rj   / m≠j
    | 3. m=j
    | 4. Rj
    | 5. Rj · ~Rj
6. m≠j

QED

Using the reflexivity rule:

1. (x)(~Gx ⊃ x≠d)   / Gd
    | 2. ~Gd   AIP
    | 3. ~Gd ⊃ d≠d   1, UI
    | 4. d=d   ID
    | 5. d≠d   3, 2, MP
    | 6. d=d · d≠d   4, 5, Conj
7. Gd

QED

An existential conclusion:

1. Rab
2. (∃x)~Rxb   /(∃x)~x=a
3. ~Rcb   2, EI
    | 4. c=a   AIP
    | 5. Rcb   1, ID
    | 6. Rcb · ~Rcb   5, 3, Conj
7. ~c=a   4-6, IP
8. (∃x)~x=a   7, EG

QED

Translate and derive:

The Faulkner scholar at Swarthmore is very learned. Therefore, all Faulkner scholars at Swarthmore are very learned.

(∃x){{(Fx · Sx) · (y)[(Fy · Sy) ⊃ x=y]} · Lx}   / (x)[(Fx · Sx) ⊃ Lx]

The argument may seem a little weird.
Remember that a definite description is definite; there is only one thing that fits the description.

1. (∃x){{(Sx · Fx) · (y)[(Sy · Fy) ⊃ x=y]} · Lx}	/ (x)[(Sx · Fx) ⊃ Lx]
│ 2. ~(x)[(Sx · Fx) ⊃ Lx]	AIP
│ 3. (∃x)~[(Sx · Fx) ⊃ Lx]	2, CQ
│ 4. ~[(Sa · Fa) ⊃ La]	3, EI
│ 5. ~[~(Sa · Fa) ∨ La]	4, Impl
│ 6. (Sa · Fa) · ~La	5, DM, DN
│ 7. {(Sb · Fb) · (y)[(Sy · Fy) ⊃ b=y]} · Lb	1, EI (to b)
│ 8. (y)[(Sy · Fy) ⊃ b=y]	7, Simp, Com, Simp
│ 9. (Sa · Fa) ⊃ b=a	8, UI (to a)
│ 10. Sa · Fa	6, Simp
│ 11. b=a	9, 10, MP
│ 12. Lb	7, Simp
│ 13. La	12, 11, ID
│ 14. ~La	6, Com, Simp
│ 15. La · ~La	13, 14, Conj
16. (x)[(Sx · Fx) ⊃ Lx]	2-15, IP
QED	

Here is a derivation or a longer argument using ID.
First we will translate:

> There are at least two cars in the driveway.
> All the cars in the driveway belong to John.
> John has at most two cars.
> So, there are exactly two cars in the driveway.

1. (∃x)(∃y)(Cx • Dx • Cy • Dy • x≠y)
2. (x)[(Cx • Dx) ⊃ Bxj]
3. (x)(y)(z)[(Cx • Bxj • Cy • Byj • Cz • Bzj) ⊃ (x=y ∨ x=z ∨ y=z)]
   / (∃x)(∃y){Cx • Dx • Cy • Dy • x≠y • (z)[(Cz • Dz) ⊃ (z=x ∨ z=y)]

The derivation follows...

1. (∃x)(∃y)(Cx • Dx • Cy • Dy • x≠y)
2. (x)[(Cx • Dx) ⊃ Bxj]
3. (x)(y)(z)[(Cx • Bxj • Cy • Byj • Cz • Bzj) ⊃ (x=y ∨ x=z ∨ y=z)]
/ (∃x)(∃y){Cx • Dx • Cy • Dy • x≠y • (z)[(Cz • Dz) ⊃ (z=x ∨ z=y)]
4. (∃y)(Ca • Da • Cy • Dy • a≠y) 1, EI
5. Ca • Da • Cb • Db • a≠b 4, EI
6. Ca • Da 5, Simp
7. (Ca • Da) ⊃ Baj 2, UI
8. Baj 7, 6, MP
9. Cb • Db 5, Simp
10. (Cb • Db) ⊃ Bbj 2, UI
11. Bbj 10, 9, MP
│ 12. ~(z)[(Cz • Dz) ⊃ (z=a ∨ z=b)] AIP
│ 13. (∃z)~[(Cz • Dz) ⊃ (z=a ∨ z=b)] 12, CQ
│ 14. (∃z)~[~(Cz • Dz) ∨ (z=a ∨ z=b)] 13, Impl
│ 15. (∃z)[(Cz • Dz) • ~(z=a ∨ z=b)] 14, DM, DN
│ 16. (∃z)(Cz • Dz • z≠a • z≠b) 15, DM
│ 17. Cc • Dc • c≠a • c≠b 16, EI
│ 18. Ca 6, Simp
│ 19. Ca • Baj 8, 18, Conj
│ 20. Cb 9, Simp
│ 21. Cb • Bbj 20, 11, Conj
│ 22. Cc • Dc 17, Simp
│ 23. (Cc • Dc) ⊃ Bcj 2, UI
│ 24. Bcj 23, 22, MP
│ 25. Cc 22, Simp
│ 26. Cc • Bcj 25, 24, Conj
│ 27. Ca • Baj • Cb • Bbj • Cc • Bcj 19, 21, 26, Conj
│ 28. (y)(z)[(Ca • Baj • Cy • Byj • Cz • Bzj) ⊃ (a=y ∨ x=z ∨ y=z)] 3, UI
│ 29. (z)[(Ca • Baj • Cb • Bbj • Cz • Bzj) ⊃ (a=b ∨ a=z ∨ b=z)] 28, UI
│ 30. (Ca • Baj • Cb • Bbj • Cc • Bcj) ⊃ (a=b ∨ a=c ∨ b=c) 29, UI
│ 31. a=b ∨ a=c ∨ b=c 30, 27, MP
│ 32. a≠b 5, Simp
│ 33. a=c ∨ b=c 31, 32, DS
│ 34. c≠a 17, Simp
│ 35. a≠c 34, ID
│ 36. b=c 33, 35, DS
│ 37. c≠b 17, Simp
│ 38. b≠c 37, ID
│ 39. b=c • b≠c 36, 38, Conj
40. (z)[(Cz • Dz) ⊃ (z=a ∨ z=b)] 12-39, IP, DN
41. Ca • Da • Cb • Db • a≠b • (z)[(Cz • Dz) ⊃ (z=a ∨ z=b)] 6, 9, 32, 40, Conj
42. (∃y){Ca • Da • Cy • Dy • a≠y • (z)[(Cz • Dz) ⊃ (z=a ∨ z=y)]} 41, EG
43. (∃x)(∃y){Cx • Dx • Cy • Dy • x≠y • (z)[(Cz • Dz) ⊃ (z=x ∨ z=y)]} 42, EG
QED

III. **Exercises**. Derive the conclusions of each of the following arguments.

1.
1. (x)(Dx ⊃ Ex)
2. Da
3. a=b / Eb

2.
1. (x)(Ax ⊃ Bx)
2. ~Bf
3. Ae / f≠e

3.
1. (x)(Hx ⊃ Jx)
2. (x)(Kx ⊃ Lx)
3. Hd · Kc
4. c=d / Jc · Ld

4.
1. (x)(y)(x=y)
2. (x)Mxx / Mab

5.
1. (x)[(∃y)Kxy ⊃ (∃z)Kzx]
2. (∃x)(Kxg · x=b) / (∃z)Kzb

6.
1. (∃x)Hx
2. (x)(y)[(Hx · Hy) ⊃ x=y] / (∃x)[Hx · (y)(Hy ⊃ x=y)]

Solutions may vary

IV. **A solution to Exercise 6**

1. (∃x)Hx	
2. (x)(y)[(Hx · Hy) ⊃ x=y]	/(∃x)[Hx · (y)(Hy ⊃ x=y)]
3. Ha	1, EI
│ 4. ~(∃x)[Hx · (y)(Hy ⊃ x=y)]	AIP
│ 5. (x)~[Hx · (y)(Hy ⊃ x=y)]	4, CQ
│ 6. (x)[~Hx ∨ ~(y)(Hy ⊃ x=y)]	5, DM
│ 7. ~Ha ∨ ~(y)(Hy ⊃ a=y)	6, UI
│ 8. ~(y)(Hy ⊃ a=y)	7, 3, DN, DS
│ 9. (∃y) ~(Hy ⊃ a=y)	8, CQ
│ 10. ~(Hb ⊃ a=b)	9, EI
│ 11. ~(~Hb ∨ a=b)	10, Impl
│ 12. Hb · ~a=b	11, DM, DN
│ 13. Hb	12, Simp
│ 14. ~a=b	12, Com Simp
│ 15. (y)[(Ha · Hy) ⊃ a=y]	2, UI
│ 16. (Ha · Hb) ⊃ a=b	15, UI
│ 17. Ha · Hb	3, 13, Conj
│ 18. a=b	16, 17, MP
│ 19. a=b · ~a=b	18, 14, Conj
20. (∃x)[Hx · (y)(Hy ⊃ x=y)]	4-19, IP

QED