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Class 40: Identity Theory, Translation II (§8.7)

I. **Special Properties of the Identity Predicate**

There is one new rule name we will use in derivations, 'ID'.
'ID' refers to three different rules governing the identity predicate.
We won't start derivations until Wednesday, but we should look at the rules, since they may help us to understand translation better.

ID Rule #1. Reflexivity: a=a

For any constant 'a', 'a' is identical to itself.
We can add, as a line in any proof, a statement of this form, as long as we refer to a specific thing.
That is, in Hurley's deductive system, reflexivity only applies to constants.

ID Rule #2. Symmetry: a=b :: b=a

Identity is commutative.
Symmetry can be applied to variables, as well as to constants.

ID Rule #3. Indiscernibility of Identicals

Consider again Superman and Clark Kent: s=c
We know that the two people are the same, so anything true of one, is true of the other.
This property is called Leibniz's law, or the Law of the Indiscernibility of Identicals:

$$(x)(y)[(x=y) \supset (\mathscr{F}x \equiv \mathscr{F}y)]$$

Written as a rule of inference, we get:

$\mathscr{F}a$
$a=b \quad / \mathscr{F}b$

If a=b, then, you may rewrite any formula containing 'a' with 'b' in the place of 'a' throughout.
Be careful not to confuse the indiscernibility of identicals, which is pretty safe, with Leibniz's contentious claim of the identity of indiscernibles.
Philosophers will recognize that the latter relies on Leibniz's contentious assertion of the Principle of Sufficient Reason.

II. **More translations, including 'exactly'**

1. Everything is identical with itself.
   $(x)x=x$

2. Nothing is distinct from itself.
   $(x){\sim}{\sim}x=x$

3. Everything is identical with something.
   $(x)(\exists y)x=y$

4. Two is the only even prime number. (t, Ex, Px, Nx)
   $Et \bullet Pt \bullet Nt \bullet (x)[(Ex \bullet Px \bullet Nx) \supset x=t]$

5. There is exactly one even prime number.
   $(\exists x)\{(Ex \bullet Px \bullet Nx) \bullet (y)[(Ey \bullet Py \bullet Ny) \supset y=x]\}$

6. There are at least two applicants.
   $(\exists x)(\exists y)[(Ax \bullet Ay) \bullet x \neq y]$

7. There are at most two applicants.
   $(x)(y)(z)[(Ax \bullet Ay \bullet Az) \supset (x=y \vee x=z \vee y=z)]$

8. There are exactly two applicants.
   $(\exists x)(\exists y)\{[Ax \bullet Ay \bullet x \neq y] \bullet (z)[Az \supset (z=x \vee z=y)]\}$

9. There are exactly three applicants.
   $(\exists x)(\exists y)(\exists z)\{[Ax \bullet Ay \bullet Az \bullet x \neq y \bullet x \neq z \bullet y \neq z] \bullet (w)[Aw \supset (w=x \vee w=y \vee w=z)]\}$

10. Adriana is a better dancer than Rene. (a, r, Bxy: x is a better dancer than y)
   Bar

11. Adriana is the best dancer.
   $(x)({\sim}x=a \supset Bax)$

12. There are exactly two dancers better than Adriana.
   $(\exists x)(\exists y)\{(Bxa \bullet Bya \bullet {\sim}x=y) \bullet (z)[Bza \supset (z=x \vee z=y)]\}$

For shorthand, we sometimes write '$(\exists nx)\mathcal{F}x$' instead of writing out the whole sentence.
So, we can write: '$(\exists 9x)Px$' for 'there are nine planets'.
But, in derivations, we would have to replace the shorthand with the full formula, including the ten quantifiers (nine existentials and one universal).
For exactly one thing, people sometimes write '$(\exists !x)\mathcal{F}x$'.

III. **Bertrand Russell's analysis for definite descriptions**:

Consider the sentence 'The king of America is bald'.
We might translate it as 'Bk'.
'Bk' is false, since there is no king of America.
So, '~Bk' should be true, since it's the negation of a false statement.
That means that 'It's false that the king of America is bald' is true.
Which seems to imply that the king of America has hair.
In fact, we want both 'The king of America is bald' and 'The king of America is not bald' to be false.
So, we had better translate the sentence differently.

'The king of America' is a definite description.
It refers to one specific object without using a name.
There are two ways to refer to an object.
We can use the name of the object, or we can describe it (e.g. the person who, the thing that)

Both sentences 'The king of America is bald' and 'The king of America is not bald' use definite descriptions to refer to an object.
They are false, due to a false presupposition in the description.
Descriptions may be complex, and we can unpack them.
'The king of America is bald' entails three simpler expressions:

A. There is a king of America.	$(\exists x)Kx$
B. There is only one king of America.	$(y)(Ky \supset y=x)$
C. That thing is bald.	Bx

Putting it all together, so that every term is within the scope of the existential quantifier, we get:

$$(\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet Bx]$$

So, the proposition is false because clause A is false.
'The king of America is not bald' is also false, for the same reason.

$$(\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet {\sim}Bx]$$

The negation only affects the third clause.
The first is still the same, and still false.
Notice that the conjunction is no longer a contradiction:

$$(\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet Bx] \bullet (\exists x)[Kx \bullet (y)(Ky \supset y=x) \bullet {\sim}Bx]$$

That conjunction is no more problematic than.

$(\exists x)Px \bullet (\exists x){\sim}Px$ e.g. Some things are purple, and some things are not purple.

Here is another example using definite descriptions:

The country called a sub-continent is India.

We can again divide it into three clauses:

A. There is a country called a sub-continent.
B. There is only one such country.
C. That country is identical with India.

So, we regiment it as: $(\exists x)\{(Cx \bullet Sx) \bullet (y)[(Cy \bullet Sy) \supset y{=}x] \bullet x{=}i\}$

Here is Russell's original example:

The author of Waverly was a genius.
$(\exists x)\{Wx \bullet (y)[Wy \supset y{=}x] \bullet Gx\}$

Consider one more

Goliath is the tallest human. (g, Hx, Txy: x is taller than y)

We can regiment it without definite descriptions:

$Hg \bullet (x)[(Hx \bullet {\sim}x{=}g) \supset Tgx]$

Or, we can focus on the definite description 'the tallest human'.

$(\exists x)[Hx \bullet (y)[(Hy \bullet {\sim}y{=}x) \supset Txy] \bullet x{=}g]$

The advantage of the latter regimentation is that it allows the elimination of constants:

$(\exists x)[Hx \bullet (y)[(Hy \bullet {\sim}y{=}x) \supset Txy] \bullet Gx]$

Remember, on the fifth exam, we had the sentence, 'Obama voted for Obama'.
I had expected:

Oo

But, some people had written:

$(\exists x)(Ox \bullet Vx)$

I had mentioned that a version using an existential quantifier is preferable to one with constants, but that we did not have enough machinery to see why, yet, or to get the version with the existential quantifier correct, yet.
The problem with the one above, is that it does not commit to a unique Obama.
We can fix that problem now:

$(\exists x)[Ox \bullet (y)(Oy \supset y{=}x) \bullet Vx]$

Still, that seems a lot more complicated than merely 'Oo'.
There are reasons to eliminate constants from regimentations.

Consider the following derivation.
It is a proof of the existence of God.

| 1. ~(∃x)x=g	Assumption, for indirect proof
| 2. (x)x=x	Principle of identity
| 3. (x)~x=g	1, Change of quantifier rule
| 4. g=g	2, UI
| 5. ~g=g	3, UI
| 6. g=g • ~g=g	4, 5, Conj
6. (∃x) x=g	1-5, Indirect proof

A moment's reflection should convince you that the same argument proves the existence of the tooth fairy.
There is a problem, here.
The problem is linked to the presence of constants within the language of first-order logic.
So, it is better to avoid them, when translating.

IV. **Exercises**

1. There is exactly one dollar bill in my wallet. (Dx, Wx)
2. The discoverer of Polonium is Polish. (Dx, Px)
3. The murderer was Colonel Mustard. (m, Mx)

V. **Solutions**

1. (∃x){(Dx • Wx) • (y)[(Dy • Wy) ⊃ y=x]}
2. (∃x){Dx • (y)(Dy ⊃ y=x) • Px]
3. (∃x)[Mx • (y)(My ⊃ y=x) • x=m]