Philosophy 240: Symbolic Logic
Fall 2008
Mondays, Wednesdays, Fridays: 9am - 9:50am

Hamilton College
Russell Marcus
rmarcus1@hamilton.edu

Supplement to Class 37: Relational Predicates, Translation II (§8.6)

I. Rules of passage

In all the discussions we have had about quantifiers, I had forgotten about prenex normal form.
We have already seen disjunctive normal form, in propositional logic, when we did the proofs of adequate sets of connectives.
In some metalogical proofs for quantificational logic, formulas must be placed in prenex normal form, in which all the quantifiers are placed in front of a formula.
In order to transform such formulas, we can use what are sometimes called rules of passage, but which are really just rules of replacement.[1]
All ten of the following rules are taken from W.V. Quine, *Methods of Logic*, Harvard University Press, 1982; though they do not appear in exactly the form that follows.
Also, in all ten transformation rules, ‘α’ stands for any formula which does not contain a free instance of the quantifier variable. (So, there’s no accidental binding, or accidental removing from binding.)

RP1:	$(\exists x)(Fx \vee Gx)$	::	$(\exists x)Fx \vee (\exists x)\, Gg$
RP2:	$(x)(Fx \bullet Gx)$	::	$(x)Fx \bullet (x)Gx$
RP3:	$(\exists x)(\alpha \bullet Fx)$	::	$\alpha \bullet (\exists x)Fx$
RP4:	$(x)(\alpha \bullet Fx)$	::	$\alpha \bullet (x)Fx$
RP5:	$(\exists x)(\alpha \vee Fx)$	::	$\alpha \vee (\exists x)Fx$
RP6:	$(x)(\alpha \vee Fx)$	::	$\alpha \vee (x)Fx$
RP7:	$(\exists x)(\alpha \supset Fx)$	::	$\alpha \supset (\exists x)Fx$
RP8:	$(x)(\alpha \supset Fx)$	::	$\alpha \supset (x)Fx$
RP9:	$(\exists x)(Fx \supset \alpha)$	::	$(x)Fx \supset \alpha$
RP10:	$(x)(Fx \supset \alpha)$	::	$(\exists x)Fx \supset \alpha$

Using the RPs, we can transform any statement of predicate logic into prenex normal form, with all the quantifiers out front.

[1] Quine notes that the rules of passage were so-called by Herbrand, in 1930, but were present in Whitehead and Russell’s *Principia Mathematica*. Prenex normal form was used by Skolem for his proof procedure, in 1922.

II. Prenex normal form

For example, consider the solution to §8.6: I.24, "If there are any cheaters, then if all referees are vigilant, they will be punished."

$(x)\{Cx \supset [(y)(Ry \supset Vy) \supset Px]\}$	
$(x)\{Cx \supset (\exists y)[(Ry \supset Vy) \supset Px]\}$	RP9
$(x)(\exists y)\{Cx \supset [(Ry \supset Vy) \supset Px]\}$	RP7

Of course, it would be unlikely that any one would translate the sentence as either of these equivalents.

An interesting fact about prenex normal form is that it is not the case that a given formula has a unique prenex form.
For example, consider Quine's sentence:

P: If there is a philosopher whom all philosophers contradict, then there is a philosopher who contradicts himself.

$(\exists x)[Fx \bullet (y)(Fy \supset Gyx)] \supset (\exists x)(Fx \bullet Gxx)$

In order to put this sentence into prenex form, we have first to change the 'x's to 'z's, so that when we stack the quantifiers in front, we won't get accidental binding.

P1: $(\exists x)[Fx \bullet (y)(Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)$

In the first set of transformations to prenex form, I will work with the 'z', then the 'y'.

	$(\exists z)(\exists x)\{[Fx \bullet (y)(Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$	by RP7
	$(\exists z)(\exists x)\{(y)[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$	by RP4
P2:	$(\exists z)(\exists x)(\exists y)\{[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$	by RP9

In the second set, I will work with the 'x', then the 'y', then the 'z'.

	$(x)\{[Fx \bullet (y)(Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$	by RP10
	$(x)\{(y)[Fx \bullet (Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$	by RP4
	$(x)(\exists y)\{[Fx \bullet (Fy \supset Gyx)] \supset (\exists z)(Fz \bullet Gzz)\}$	by RP9
P3:	$(x)(\exists y)(\exists z)\{[Fx \bullet (Fy \supset Gyx)] \supset (Fz \bullet Gzz)\}$	by RP7

P2 and P3 are equivalent to P1.
P2 and P3 are both in prenex form.
But, they differ in form from each other.

There are (I think) two other prenex forms equivalent to P1.
See if you can work them out.

III. Applying prenex form to our earlier discussion

Consider again the three statements we discussed earlier.

35. (∃x)[Px • (y)(Qy ⊃ Rxy)]
36. (∃x)(y)[Px • (Qy ⊃ Rxy)]
37. (∃x)(y)[Px ⊃ (Qy ⊃ Rxy)]

35 and 36 are shown equivalent by RP4. (Ignore the existential quantifier in front.)
No similar transformation is admissible between 37 and 35.
Using RP8, we can transform 37 into:

38. (∃x)[Px ⊃ (y)(Qy ⊃ Rxy)]

I had said that you can not just move quantifiers around, and offered the inequivalence of 13 and 13' as evidence:

13. (x)[(∃y)Lxy ⊃ Hx]
13'. (x)(∃y)(Lxy ⊃ Hx)

Now, we can see that 13 is equivalent to:

39. (x)(y)(Lxy ⊃ Hx) by RP10.

This transformation feels right, since both 13 and 39 can be interpreted as, "If anyone loves someone, then s/he is happy."
On the other hand, 13' is equivalent to:

40. (x)[(y)Lxy ⊃ Hx] by RP9.

That transformation strikes me as strange.
It might even make me call RP9 into question, if I didn't notice the following:

1. (x)(∃y)(Lxy ⊃ Hx)	13'
2. (x)(∃y)(~Lxy ∨ Hx)	1, Impl
3. (x)(∃y)(Hx ∨ ~Lxy)	2, Com
4. (x)[Hx ∨ (∃y)~Lxy]	3, RP5
5. (x)[(∃y)~Lxy ∨ Hx]	4, Com
6. (x)[~(y)Lxy ∨ Hx]	5, CQ
7. (x)[(y)Lxy ⊃ Hx]	6, Impl, which is 40.

See if you can wrap your heads around that transformation.
Also, if you want something sort of fun to do, see if you can determine the relations among:

(∃x)(α ≡ Fx)
α ≡ (∃x)Fx
(x)(α ≡ Fx)
α ≡ (x)Fx