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Class 37: Relational Predicates, Translation II (§8.6)

I. **Translation with relations**

We worked on translation using relations on the Wednesday before break.

1. Jen reads all books written by Asimov. (Bx: x is a book; Wxy: x writes y; Rxy: x reads y; j: Jen; a: Asimov)
   (x)[(Bx • Wax) ⊃ Rjx]
2. Some people read all books written by Asimov.
   (∃x){Px • (y)[(By • Way) ⊃ Rxy]}
3. Some people read all books written by some one.
   (∃x){Px • (y){[By • (∃z)(Pz • Wzy)] ⊃ Rxy}}

The next three sentences were on the previous handout, and involve triadic predicates.

4. Everyone buys something from some store. (Bxyz: x buys y from z)
   (x)[Px ⊃ (∃y)(∃z)(Sz · Bxyz)]
5. There is a store from which everyone buys something.
   (∃x){Sx · (y)[Py ⊃ (∃z)Byzx]}
6. No store has everyone for a customer.
   ~(∃x){Sx · (y)[Py ⊃ (∃z)Byzx]} or (x){Sx ⊃ (∃y)[Py · (z)~Byzx]}

There were questions on the Wednesday before break about the scope of our quantifiers, and whether we can bring them all outside (i.e. give them wide scope) or whether, as I advised, we only introduce the quantifiers when needed (i.e. give them narrow scope).
Mostly, it is good form to keep narrow scope.
On occasion, we will just put all quantifiers in front, using broad scope.
But, we must be careful.
One reason to keep scope narrow is just to help keep track of the order of quantifiers.
Sometimes the order of the quantifiers and the scope doesn't matter:
'Everyone loves everyone' can be written as any of the following:

7. (x)[Px ⊃ (y)(Py ⊃ Lxy)]
7'. (x)(y)[(Px · Py) ⊃ Lxy]
7". (y)(x)[(Px · Py) ⊃ Lxy]

Technically, 7" is 'everyone is loved by everyone'.
But all three statements are logically equivalent.
Similarly, 'someone loves someone' can be written as any of the following:

8. (∃x)[Px · (∃y)(Py · Lxy)]
8'. (∃x)(∃y)[(Px · Py) · Lxy]
8". (∃y)(∃x)[(Px · Py) · Lxy]

8'' is ‘someone is loved by someone’.
Again, these are all equivalent.

When you mix universals with existentials, you have to be careful, since reversing the order of the quantifiers changes the meaning of the proposition.
None of the following examples are equivalent:

9. $(x)(\exists y)[(Px \cdot Py) \cdot Lxy]$
   Everyone loves someone.
10. $(\exists y)(x)[(Px \cdot Py) \supset Lxy]$
   Someone is loved by everyone.
11. $(\exists x)(y)[(Px \cdot Py) \supset Lxy]$
   Someone loves everyone.
12. $(y)(\exists x)[(Px \cdot Py) \cdot Lxy]$
   Everyone is loved by someone.

Note that the first word in each translation above corresponds to the leading quantifier.
Also, note the main connective is determined by the innermost quantifier.
If the innermost quantifier is existential, the main connective is a conjunction.
If the innermost quantifiers is universal, the main connective is a conditional.
This may be clearer if we take the quantifiers inside.

9'. $(x)[Px \supset (\exists y)(Py \cdot Lxy)]$
10'. $(\exists x)[Px \cdot (y)(Py \supset Lyx)]$
11'. $(\exists x)[Px \cdot (y)(Py \supset Lxy)]$
12'. $(x)[Px \supset (\exists y)(Py \bullet Lyx)]$

Some people in class expressed the thought that moving the quantifiers out front would not change the meaning of a proposition, as long as one did not bind any further instances of the quantifier variable.
But consider:

13. $(x)[(\exists y)Lxy \supset Hx]$

is not equivalent to:

13'. $(x)(\exists y)(Lxy \supset Hx)$

Here we have moved the existential quantifier out front, and merely brought the ‘Qx’ into the scope of ‘$(\exists y)$’, which does not bind it.
13 can be interpreted as ‘All lovers are happy’, or ‘for any x, if there is a y that x loves, then x is happy’.
In that case, 13' would be ‘For any x, there is a y such that if x loves y then x is happy’.
13 does not commit to the existence of something that, by being loved, makes a person happy.
13' does.
Consider the universe in which there are things that can never be happy, for which nothing could make them happy.
13 could still be true, but 13' would have to be false.

II. **Logical truths in quantificational logic, including overlapping quantifiers**

Let's look at some equivalences in quantificational logic, and some statements that are not equivalent. Many of the following equivalences can easily be proved by two instances conditional proof:

14. $(x)Fx \bullet (x)Gx \;::\; (x)(Fx \bullet Gx)$

Additionally,

15. $(x)Fx \vee (x)Gx \;\vdash\; (x)(Fx \vee Gx)$

'$\alpha \vdash \beta$' means that β can be derived from α.
'$\vdash$' is the meta-linguistic form of '$\supset$'
Its negation is normally written with a slash through it, but I don't have easy access to that symbol, so I will write '$\sim\vdash$'.
The converse of the last entailment does not hold:

16. $(x)(Fx \vee Gx) \;\sim\vdash\; (x)Fx \vee (x)Gx$

To see it fail, just substitute 'P' for F and '~P' for G
Similarly, this entailment holds:

17. $(\exists x)(Fx \bullet Gx) \;\vdash\; (\exists x)Fx \bullet (\exists x)Gx$

But its converse does not:

18. $(\exists x)Fx \bullet (\exists x)Gx \;\sim\vdash\; (\exists x)(Fx \bullet Gx)$

The following equivalence holds:

19. $(x)(Fx \bullet \alpha) \;::\; (x)Fx \bullet \alpha$
    e.g. $(x)[Px \bullet (\exists y)Qy] \;::\; (x)Px \bullet (\exists y)Qy$

In the above instance, the scope of the universal quantifier does not matter, as long as the α does not contain any free instances of the quantifier variable ('x' in these cases).
Similarly:

20. $(x)(Fx \bullet \alpha) \;::\; (x)Fx \bullet \alpha$
    e.g. $(x)[Px \bullet (\exists y)Qy] \;::\; (x)Px \bullet (\exists y)Qy$

and

21. $(x)(\alpha \supset Fx) \;::\; \alpha \supset (x)Fx$
    e.g. $(x)[(\exists y)Py \supset Qx)] \;::\; (\exists y)Py \supset (x)Qx$

This following formulae are also equivalent:

22. $(\exists x)(\alpha \supset \mathcal{F}x) \;::\; \alpha \supset (\exists x)\mathcal{F}x$
    e.g. $(\exists x)[(y)Py \supset Qx)] \;::\; (y)Py \supset (\exists x)Qx$

But though this entailment holds:

23. $(x)(\mathcal{F}x \supset \alpha) \vdash (x)\mathcal{F}x \supset \alpha$
    e.g. $(x)[Px \supset (\exists y)Qy] \vdash (x)Px \supset (\exists y)Qy$

Its converse does not.

24. $(x)\mathcal{F}x \supset \alpha \;{\sim}{\vdash}\; (x)(\mathcal{F}x \supset \alpha)$
    e.g. $(x)Px \supset (\exists y)Qy \;{\sim}{\vdash}\; (x)[Px \supset (\exists y)Qy]$

The next equivalence might be counter-intuitive:

25. $(x)(\mathcal{F}x \supset \alpha) \;::\; (\exists x)\mathcal{F}x \supset \alpha$
    e.g. $(x)[Px \supset (\exists y)Qy] \;::\; (\exists x)Px \supset (\exists y)Qy$

So let's take a moment to prove it.
Consider first what happens when α is true, and then when α is false.
If α is true, then both formulas will turn out to be true.
    '$\mathcal{F}x \supset \alpha$' will be true for every instance of x, since the consequent is true.
    So, the universal generalization of each such formula (which is the formula on the left) will be true.
    Similarly, the consequent of the formula on the right is just α, so if α is true, the whole formula will be true.
If α is false, then the truth value of each formula will depend.
    If the formula on the left turns out to be true, it must be because '$\mathcal{F}x$' is false, for every x.
    But then, '$(\exists x)\mathcal{F}x$' will be false, and so the formula on the right turns out to be true.
    If the formula on the right turns out to be true, then it must be because '$(\exists x)\mathcal{F}x$' is false.
    And so, there will be no value of 'x' that makes '$\mathcal{F}x$' true, and so the formula on the right will also turn out to be (vacuously) true.

25 allows us to translate 'If anything was damaged, then everyone gets upset' in two ways:

26. $(\exists x)Dx \supset (x)(Px \supset Ux)$
27. $(x)[Dx \supset (y)(Py \supset Uy)]$

That is, 26 is logically equivalent to 27.
Similarly, this entailment holds:

28. $(\exists x)\mathcal{F}x \supset \alpha \vdash (\exists x)(\mathcal{F}x \supset \alpha)$
    e.g. $(\exists x)Px \supset (\exists y)Qy \vdash (\exists x)[Px \supset (\exists y)Qy]$

But, the right side is hard to comprehend.

The converse of 28 does not hold:

29. $(\exists x)(\mathcal{F}x \supset \alpha) \;\sim\vdash\; (\exists x)\mathcal{F}x \supset \alpha$
e.g. $(\exists x)[Px \supset (\exists y)Qy] \;\sim\vdash\; (\exists x)Px \supset (\exists y)Qy$

On the other hand, these two are equivalent:

30. $(\exists x)(\mathcal{F}x \supset \alpha) :: (x)\mathcal{F}x \supset \alpha$
e.g. $(\exists x)[Px \supset (\exists y)Qy] \;\vdash\; (x)Px \supset (\exists y)Qy$

Each of which almost make sense.
Here are four logical truths, in the object language:

31. $(y)[(x)Fx \supset Fy]$
32. $(y)[Fy \supset (\exists x)Fx]$
33. $(\exists y)[Fy \supset (x)Fx]$
34. $(\exists y)[(\exists x)Fx \supset Fy]$

These should all be provable using IP.
Note that each one has a similarity to one of the four rules for removing or replacing quantifiers.

III. **Scope**

We started looking into these logical truths because we were interested in questions of scope.
In particular, I believe that we had a sentence of the following form:

35. $(\exists x)[Px \bullet (y)(Qy \supset Rxy)]$

It was suggested that 35 is logically equivalent to

36. $(\exists x)(y)[Px \bullet (Qy \supset Rxy)]$

I agree, though I think 36 is just really weird, which is why I hesitated in class.
The proofs of the equivalence are below, in the appendix to these notes.
I had sketched at least one direction of the proof in class, but I wasn't satisfied at that point.
There is a restriction on UG, when there are overlapping quantifiers.
We will look at that restriction on Wednesday.
I hadn't remembered the details of the restriction, and so I had to think about it a bit more.
I had thought that 35 might be equivalent to:

37. $(\exists x)(y)[Px \supset (Qy \supset Rxy)]$

37 looks better to me, since the '$\supset$' is the main connective inside, and it corresponds to the innermost quantifier.
But, 37 is not logically equivalent to 35.
In fact, 35 $\vdash$ 37, but 37 $\sim\vdash$ 35.

Again, see the appendix for 35 $\vdash$ 37.
To see that 37 $\sim\vdash$ 35, we can construct a counter-example in a universe with two-members
I'll expand 37 in two steps, first removing the existential quantifier, then the universal.

37'. (y)[Pa ⊃ (Qy ⊃ Ray)] ∨ (y)[Pb ⊃ (Qy ⊃ Rby)]
37". {[Pa ⊃ (Qa ⊃ Raa)] • [Pa ⊃ (Qb ⊃ Rab)]} ∨ {[Pb ⊃ (Qa ⊃ Rba)] • [Pb ⊃ (Qb ⊃ Rbb)]}

I'll do the same for 35:

35'. [Pa • (y)(Qy ⊃ Ray)] ∨ [Pb • (y)(Qy ⊃ Rby)]
35". [Pa • (Qa ⊃ Raa) • (Qb ⊃ Rab)] ∨ [Pb • (Qa ⊃ Rba) • (Qb ⊃ Rbb)]

To form the counter-example, just assign false to both 'Pa' and 'Pb'.
Then, both conjuncts in 35" are false, but all the conditionals in 37" are (vacuously) true.

IV. **Exercises**. Translate each of the following sentences into predicate logic.

1. Everyone loves something. (Px, Lxy)
2. No one knows everything. (Px, Kxy)
3. No one knows everyone.
4. Every woman is stronger than some man. (Wx, Mx, Sxy: x is stronger than y)
5. No cat is smarter than any horse. (Cx, Hx, Sxy: x is smarter than y)
6. Dead men tell no tales. (Dx, Mx, Tx, Txy: x tells y)
7. There is a city between New York and Washington. (Cx, Bxyz: y is between x and z)
8. Everyone gives something to someone. (Px, Gxyz: y gives x to z)
9. A dead lion is more dangerous than a live dog. (Ax: x is alive, Lx, Dx, Dxy: x is more dangerous than y)
10. A lawyer who pleads his own case has a fool for a client. (Lx, Fx, Pxy: x pleads y's case; Cxy: y is a client of x)

V. **Solutions**

1. (x)[Px ⊃ (∃y)Lxy]
2. (x)[Px ⊃ (∃y)~Kxy] or ~(∃x)[Px · (y)Kxy]
3. (x)[Px ⊃ (∃y)(Py · ~Kxy)] or ~(∃x)[Px · (y)(Py ⊃ Kxy]
4. (x)[Wx ⊃ (∃y)(My · Sxy)]
5. ~(∃x)[Cx · (∃y)(Hy · Sxy)] or (x)[Cx ⊃ (y)(Hy ⊃ ~Sxy)]
6. (x)[(Dx · Mx) ⊃ ~(∃y)(Ty · Txy)]
7. (∃x)(Cx · Bnxw)
8. (x)[Px ⊃ (∃y)(∃z)(Pz · Gyxz)]
9. (x){(~Ax · Lx) ⊃ (y)[(Ay · Dy) ⊃ Dxy]}
10. (x)[(Lx · Pxx) ⊃ (∃y)(Fy · Cxy)] or (x)[(Lx · Pxx) ⊃ Fx]

Note that these two translations aren't equivalent.
The first translates the surface grammar.
The second translates the meaning.

VI. **Appendix**

$35 \vdash 36$

1. $(\exists x)[Px \bullet (y)(Qy \supset Rxy)]$	
2. $Pa \bullet (y)(Qy \supset Ray)$	1, EI
$\vert$ 3. Qy	ACP
$\vert$ 4. $(y)(Qy \supset Ray)$	2, Com, Simp
$\vert$ 5. $Qy \supset Ray$	4, UI
$\vert$ 6. Ray	5, 3, MP
7. $Qy \supset Ray$	3-6, CP
8. Pa	2, Simp
9. $Pa \bullet (Qy \supset Ray)$	8, 7, Conj
10. $(y)[Pa \bullet (Qy \supset Ray)]$	9, UG
11. $(\exists x)(y)[Px \bullet (Qy \supset Rxy)]$	10, EG

QED

$36 \vdash 35$

1. $(\exists x)(y)[Px \bullet (Qy \supset Rxy)]$	
2. $(y)[Pa \bullet (Qy \supset Ray)]$	1, EI
3. $Pa \bullet (Qy \supset Ray)$	2, UI
4. $Qy \supset Ray$	3, Com, Simp
5. $(y)(Qy \supset Ray)$	4, UG
6. Pa	3, Simp
7. $Pa \bullet (y)(Qy \supset Ray)$	6, 5, Conj
8. $(\exists x)[Px \bullet (y)(Qy \supset Rxy)$	7, EG

QED

$35 \vdash 37$

1. $(\exists x)[Px \bullet (y)(Qy \supset Rxy)]$	
$\vert$ 2. $\sim(\exists x)(y)[Px \supset (Qy \supset Rxy)]$	AIP
$\vert$ 3. $(x)(\exists y)\sim[Px \supset (Qy \supset Rxy)]$	2, CQ
$\vert$ 4. $(x)(\exists y)\sim[\sim Px \vee \sim Qy \vee Rxy]$	3, Impl, Impl
$\vert$ 5. $(x)(\exists y)(Px \bullet Qy \bullet \sim Rxy)$	4, DM, DN
$\vert$ 6. $Pa \bullet (y)(Qy \supset Ray)$	1, EI
$\vert$ 7. $(\exists y)(Pa \bullet Qy \bullet \sim Ray)$	5, UI
$\vert$ 8. $Pa \bullet Qb \bullet \sim Rab$	7, EI
$\vert$ 9. $(y)(Qy \supset Ray)$	6, Com, Simp
$\vert$ 10. $Qb \supset Rab$	9, UI
$\vert$ 11. Qb	8, Com, Simp
$\vert$ 12. Rab	10, 11, MP
$\vert$ 13. $\sim Rab$	8, Com, Simp
$\vert$ 14. $Rab \bullet \sim Rab$	12, 13, Conj
15. $(\exists x)(y)[Px \supset (Qy \supset Rxy)]$	2-14, IP, DN

QED