Philosophy 240: Symbolic Logic | Hamilton College
Fall 2008 | Russell Marcus
Mondays, Wednesdays, Fridays: 9am - 9:50am | rmarcus1@hamilton.edu

Class 34: Predicate Invalidity (§8.5)

I. Invalidity we have seen

Recall how we proved invalidity in propositional logic.
Consider an argument:

1. A ⊃ B
2. ~(B · A) /A ≡ B

We lined up the premises and conclusion

A ⊃ B / ~(B · A) // A ≡ B

We then assigned truth values to the component sentences to form a counterexample.
A counterexample is a valuation which makes the premises true and the conclusion false

A	⊃	B	/	~	(B	·	A)	//	A	≡	B
F	**T**	T		**T**	T	F	F		F	**F**	T

So, the argument is shown invalid when A is false and B is true.

Also, recall that we used an informal counter-example method to prove arguments invalid.
Consider the following interpretations of the component sentences of the above argument:

A: November is warm in Clinton.
B: The Phillies won the World Series this year.

Under this interpretation, the premises of the argument are both true, but the conclusion is false.

II. The informal counter-example method

In predicate logic, we can use the informal method, too.
Consider:

(x)(Wx ⊃ Hx)
(x)(Ex ⊃ Hx) / (x)(Wx ⊃ Ex)

We can provide an interpretation of the predicates that yields true premises but a false conclusion.

Wx: x is a whale
Ex: x is an elephant
Hx: x is heavy

So, 'all whales are heavy' and 'all elephants are heavy' are both true.
But, 'all whales are elephants' is false.

III. Exercises A

Show invalid, using the counterexample method:

1.
1. $(x)(Ax \supset Bx)$
2. Bj $\quad$ / Aj

2.
1. $(\exists x)(Ax \cdot Bx)$
2. Aa $\quad$ / Ba

3.
1. $(x)(Hx \supset Ix)$
2. $(x)(Hx \supset {\sim}Jx)$ $\quad$ / $(x)(Ix \supset {\sim}Jx)$

IV. The method of finite universes

The informal counter-example method is fine for shorter, simpler arguments.
Some of you are probably smart enough to come up with something for an argument like:

1. $(x)[Ux \supset (Tx \supset Wx)]$
2. $(x)[Tx \supset (Ux \supset {\sim}Wx)]$
3. $(\exists x)(Ux \cdot Wx)$

$\therefore (\exists x)(Ux \cdot Tx)$

But, it would be nice to have a method which requires less ingenuity.
If an argument is valid, then it is valid, no matter what the interpretation of the constants, and no matter what exists in the world.
Logical truths are true in all possible universes.
They have no prejudices about what exists in the world.
If they did, then they wouldn't be logical truths, but empirical truths!
Even if the world has only one member, or two or three, valid arguments should be valid.

Consider the following invalid argument:

$(x)(Wx \supset Hx)$
$(x)(Ex \supset Hx)$ $\quad$ / $(x)(Wx \supset Ex)$

Imagine there were only one object in the universe. Let's call it 'a'.
Then:

$(x)(Wx \supset Hx)$	would be equivalent to	$Wa \supset Ha$
$(x)(Ex \supset Hx)$	would be equivalent to	$Ea \supset Ha$
$\therefore (x)(Wx \supset Ex)$	would be equivalent to	$Wa \supset Ea$

Now, assign truth values, as in the propositional case to make the premises true and the conclusion false:

Wa	⊃	Ha	/	Ea	⊃	Ha	//	Wa	⊃	Ea
T	**T**	T		F	**T**	T		T	**F**	F

So, the argument is shown invalid in a one-member universe, where Wa is true, Ha is true, and Ea is false.

This method works for complex arguments, as well.
Consider the argument from the beginning of this section.

1. (x)[Ux ⊃ (Tx ⊃ Wx)]
2. (x)[Tx ⊃ (Ux ⊃ ~Wx)]
3. (∃x)(Ux · Wx)

∴(∃x)(Ux · Tx)

Ua	⊃	(Ta	⊃	Wa)	/	Ta	⊃	(Ua	⊃	~	Wa)	/	Ua	•	Wa	//	Ua	•	Ta
T	**T**	F	T	T		F	**T**	T	F	F	T		T	**T**	T		T	**F**	F

The argument is shown invalid in a one-member universe, where Ua is true; Ta is false; and Wa is true.

Not all invalid arguments are shown invalid in a one-member universe.
All we need is one universe in which they are shown invalid, to show that they are invalid.
Even if an argument has no counter-example in a one-member universe, it might still be invalid!

V. Universes of more than one member

Consider the following invalid argument:

(x)(Wx ⊃ Hx)
(∃x)(Ex · Hx) / (x)(Wx ⊃ Ex)

In a one-object universe, we have:

Wa	⊃	Ha	/	Ea	·	Ha	//	Wa	⊃	Ea
				F				T	**F**	F

There is no way to construct a counterexample.
But the argument is invalid.
(I know, because I made it!)
We have to consider a larger universe.

If there are two objects in a universe, a and b:

$(x)\mathscr{F}x$ becomes $\mathscr{F}a \cdot \mathscr{F}b$ because every object has $\mathscr{F}$
$(\exists x)\mathscr{F}x$ becomes $\mathscr{F}a \vee \mathscr{F}b$ because only some objects have $\mathscr{F}$

If there are three objects in a universe, then

$(x)\mathscr{F}x$ becomes $\mathscr{F}a \cdot \mathscr{F}b \cdot \mathscr{F}c$
$(\exists x)\mathscr{F}x$ becomes $\mathscr{F}a \vee \mathscr{F}b \vee \mathscr{F}c$

Returning to the problem...
In a universe of two members, we represent the argument is equivalent to:

$(Wa \supset Ha) \cdot (Wb \supset Hb)$ / $(Ea \cdot Ha) \vee (Eb \cdot Hb)$ // $(Wa \supset Ea) \cdot (Wb \supset Eb)$

Now, assign values to each of the terms to construct a counterexample.

(Wa	⊃	Ha)	·	(Wb	⊃	Hb)	/	(Ea	·	Ha)	∨	(Eb	·	Hb)
T	T	T	**T**	F	T	T		F	F	T	**T**	T	T	T

//	(Wa	⊃	Ea)	·	(Wb	⊃	Eb)
	T	F	F	**F**	F	T	T

The argument is shown invalid in a two-member universe, when

Wa: true Wb: false
Ha: true Hb: true
Ea: false Eb: true

The assignment of values to each term is called a counter-example, even though we are using the method of finite universes.

. Constants

When expanding formulas into finite universes, constants get rendered as themselves.
That is, we don't expand a term with a constant when moving to a larger universe.
Consider:

(∃x)(Ax · Bx)
Ac /Bc

We can't show it invalid in a one-member universe.

Ac	·	Bc	/	Ac	//	Bc
	F	F				F

We must move to a two-member universe.
Here, we generate a counter-example.

(Ac	·	Bc)	∨	(Aa	·	Ba)	/	**Ac**	//	**Bc**
T	F	F	**T**	T	T	T		**T**		**F**

This argument is shown invalid in a two-member universe, when
Ac: true Bc: false
Aa: true Ba: true
Some arguments need three, four, or even infinite models to be shown invalid.

VII. Propositions whose main connective is not a quantifier

Consider the following argument:

(∃x)(Px · Qx)
(x)Px ⊃ (∃x)Rx
(x)(Rx ⊃ Qx) /(x)Qx

In a one-member universe, this argument gets rendered as:

Pa · Qa / Pa ⊃ Ra / Ra ⊃ Qa // Qa

But there is no counter-example in a one-member universe.

Pa	·	Qa	/	Pa	⊃	Ra	/	Ra	⊃	Qa	//	**Qa**
	F	F										**F**

n a two-member universe, note what happens to the second premise:

(Pa · Qa) ∨ (Pb · Qb) / (Pa · Pb) ⊃ (Ra ∨ Rb) / (Ra ⊃ Qa) · (Rb ⊃ Qb) // Qa · Qb

Each quantifier is unpacked independently.
The main connective, the conditional, remains the main connective.
We can clearly see here the difference between instantiation and translation into a finite universe.

We can construct a counterexample for this argument in a two-member universe:

(Pa	·	Qa)	∨	(Pb	·	Qb)	/	(Pa	·	Pb)	⊃	(Ra	∨	Rb)
	F	F	**T**	T	T	T				T	**T**	F	T	T

/	(Ra	⊃	Qa)	·	(Rb	⊃	Qb)	//	Qa	·	Qb
	F	T	F	**T**	T	T	T		F	F	T

This argument is shown invalid in a two-member universe, when
Pa: either true or false Pb: true
Qa: false Qb: true
Ra: false Rb: true

(There is another solution. Can you construct it?)

III. Exercises B. Show each of the following arguments invalid by generating a counter-example using the method of finite universes.

1.
1. $(x)(Ex \supset Fx)$
2. $(\exists x)(Gx \cdot {\sim}Fx)$ / $(\exists x)(Ex \cdot {\sim}Gx)$

2.
1. $(x)(Bx \supset {\sim}Dx)$
2. ${\sim}Bj$ / Dj

3.
1. $(x)(Hx \supset {\sim}Ix)$
2. $(\exists x)(Jx \cdot {\sim}Ix)$ / $(x)(Hx \supset Jx)$

4.
1. $(x)(Kx \supset {\sim}Lx)$
2. $(\exists x)(Mx \cdot Lx)$ / $(x)(Kx \supset {\sim}Mx)$

5.
1. $(\exists x)(Px \cdot Qx)$
2. $(x)(Qx \supset {\sim}Rx)$
3. Pa / $(x){\sim}Rx$

6.
1. $(x)(Ax \supset Bx)$
2. $(\exists x)(Dx \cdot Bx)$
3. $(\exists x)(Dx \cdot {\sim}Bx)$ / $(x)(Ax \supset Dx)$

IX. Solutions

Sample answers to Exercises A
1. Ax: x is an apple; Bx: x is a fruit; j: a pear
2. Ax: x is a Met; Bx: x is a pitcher; a: Carlos Delgado
3. Hx: x is a desk; Ix: x has legs; Jx: x has arms

Sample answers to Exercises B
1. Shown invalid in a one-member universe, where Ga: true; Ea: false; Fa: false
2. Shown invalid in a one-member universe, where Bj: false; Dj: false
3. Shown invalid in a two-member universe, where Ha: true; Ia: false; Ja: false; Hb: true or false; Ib: false; Jb: true
4. Shown invalid in a two-member universe, where Ka: false; La: true; Ma: true; Kb: true; Lb: false; Mb: true.
5. Shown invalid in a two-member universe, where Pa: true; Qa: false; Ra: true; Pb: true; Qb: true; Rb: false
6. Shown invalid in a three-member universe, where Aa: true; Ba: true; Da: false; Ab: true or false; Bb: true; Db: true; Ac: false; Bc: false; Dc: true