Class 27: Adequate Sets of Connectives

I. Eliminating the biconditional and conditional.

We have seen how to get rid of the biconditional by defining it in terms of the conditional.
This was the rule we called material equivalence.
Call a connective superfluous if it can be defined in terms of other connectives.

Theorem 1: The biconditional is superfluous.
Proof 1A: We just need to show that 'α ≡ β' and '(α ⊃ β) • (β ⊃ α)' are logically equivalent.
We can do this by method of truth tables.

α	≡	β
T	**T**	T
T	**F**	F
F	**F**	T
F	**T**	F

(α	⊃	β)	·	(β	⊃	α)
T	T	T	**T**	T	T	T
T	F	F	**F**	F	T	T
F	T	T	**F**	T	F	F
F	T	F	**T**	F	T	F

QED

Notice that the conditional is superfluous also, according to the rule of material implication.

Theorem 2: The conditional is superfluous.
Proof 2A: By method of truth tables.

α	⊃	β
T	**T**	T
T	**F**	F
F	**T**	T
F	**T**	F

~	α	∨	β
F	T	**T**	T
F	T	**F**	F
T	F	**T**	T
T	F	**T**	F

Proof 2B: By method of conditional proof.
To show that two statements are logically equivalent, we show that each entails the other.

Assume: 'α ⊃ β'.
Derive: '~α ∨ β'.
Then, assume '~α ∨ β'.
Derive: 'α ⊃ β'.

QED

Notice that we can prove Theorem 1 by the method in Proof 2B.

Proof 1B: Assume '$\alpha \equiv \beta$'.
Derive '$(\alpha \supset \beta) \bullet (\beta \supset \alpha)$'.
Assume '$(\alpha \supset \beta) \bullet (\beta \supset \alpha)$'.
Derive '$\alpha \equiv \beta$'.

In fact, reading those sentences should convince you of the legitimacy of this method of proof itself!
We have on our hands two distinct notions of logical equivalence.
LE_1: Two statements are logically equivalent iff they have the same values in every row of the truth table.
LE_2: Two statements are logically equivalent iff each is derivable, using our system of deduction, from the other.
We hope that LE_1 and LE_2 yield the same results.
To prove that LE_1 and LE_2 yield the same results, we have to justify our system of deduction.
That work is left for another occasion.

Combining Theorems 1 and 2, we discover that any sentence which can be written as a biconditional can be written in terms of negation, conjunction, and disjunction.
We have two methods of showing that any sentence which is translated as a biconditional can be just written in terms of the negation, conjunction, and disjunction.
Consider: 'Dogs bite if and only if they are startled'.

We regiment it directly as a biconditional:	$B \equiv S$
We eliminate the biconditional:	$(B \supset S) \bullet (S \supset B)$
And, finally, eliminate the conditional:	$({\sim}B \vee S) \bullet ({\sim}S \vee B)$

Two questions arise:
Q1. How can we be sure that all sentences can be written with just the five connectives?
Q2. Can we get rid of more connectives? What is the fewest number of connectives that we need?

We will answer both questions, here.

II. Defining adequacy and disjunctive normal form

A set of connectives is called **adequate** iff corresponding to every possible truth table there is at least one sentence using only those connectives.
By "every possible truth table," I mean every combination of 'T's and 'F's in the column under the major operator.
To give you a taste of what we are after, consider a severely limited adequacy result.

Theorem 3: Negation and conjunction are adequate, if we use only one propositional variable.
Proof 3: By sheer force.
There are only four possible truth tables: TT, TF, FT, FF
Here are statements for each of them.

~	(α	•	~	α)
T	T	F	F	T
T	F	F	T	F

α
T
F

~	α
F	T
T	F

α	•	~	α
T	**F**	F	T
F	**F**	T	F

QED

We want to demonstrate the general theorem that the five connectives are adequate for any number of propositional variables.
By Theorems 1 and 2, we know that the five connectives are adequate if, and only if, the three (negation, conjunction, and disjunction) are adequate.
In order to prove the general theorem, consider Disjunctive Normal Form (DNF).
A sentence is in DNF iff if is a series of disjunctions, each disjunct of which is a conjunction of simple letters or negations of simple letters.
A single letter or its negation can be considered a degenerate conjunction or disjunction.
(For the purposes of this exercise, we can drop brackets among three or more conjuncts, or three or more disjuncts, though we can not mix and match.)

Exercises A: Pick out which of the following sentences are in DNF:

1. (P • ~Q) ∨ (P • Q)
2. (P • Q • R) ∨ (~P • ~Q • ~R)
3. ~P ∨ Q ∨ R
4. (P ∨ Q) • (P ∨ ~R)
5. (P • Q) ∨ (P • ~Q) ∨ (~P • Q) ∨ (~P • ~R)
6. (~P • Q) • (P • R) ∨ (Q • ~R)
7. (P • ~Q • R) ∨ (Q • ~R) ∨ ~Q
8. ~(P • Q) ∨ (P • R)
9. P • Q
10. ~P

III. Proving adequacy and inadequacy for familiar connectives

Theorem 4: The set of negation, conjunction, and disjunction {~, •, ∨} is adequate.
Proof 4: By cases.
For any size truth table, with any number of connectives, there are three possibilities for the column under the major operator.
Case 1: Every row is false.
Case 2: There is one row which is true, and every other row is false.
Case 3: There is more than one row which is true.

Case 1:
Construct a sentence with one variable in the sentence conjoined with its negation and each of the remaining variables.
So, if you have variables P, Q, R, S, and T, you would write:

(P • ~P) • (Q • S • T)

If you have more variables, add more conjuncts.
The resulting formula, in DNF, is false in every row, and uses only conjunction and negation.

Case 2:
Consider the row in which the statement is true.
Write a conjunction of the following statements:

For each variable, if it is true in that row, write that variable.
For each variable, if it is false in that row, write the negation of that variable.

The resulting formula is in DNF (the degenerate disjunction) and is true in only the prescribed row.

Example:
Consider a formula with two variables: P=TTFF; Q=TFTF;
Major operator=FFFT
We consider the last row only, in which P and Q are both false.
So, we get: '~P • ~Q'.
Note that this formula is in DNF.
Also, note that it is equivalent to a different statement, '~(P ∨ Q)', by DeMorgan's Law.
There are multiple formulas which will yield the same truth table.
In fact, there are infinitely many ways to produce each truth table.
(Consider, one can always just add pairs of tildes to a formula.)

Case 3:
For each row in which the statement is true, perform the method described in Case 2.
Then, form the disjunction of all the resulting formulas.

Example:
Consider a formula with three variables.
P=TTTTFFFF; Q=TTFFTTFF; R=TFTFTFTF
Major operator=TFFTFFFF
To construct a formula with that truth table, consider only the first and fourth rows.
(P • Q • R) ∨ (P • ~Q • ~R)
(Punctuation can easily be added to make the formula well-formed.)
QED

Given Theorem 4, and the methods used in the proofs of Theorems 1 and 2, we can prove several other sets of connectives adequate.

Theorem 5: The set {∨, ~} is adequate.
Proof 5:

By Theorem 4, we can write a formula for any truth table using as connectives only those in the set {∨, •, ~}.
'α • β' is equivalent to '~(~α ∨ ~α)'.
So, we can replace any occurrence of '•' in any formula, according to the above equivalence.

QED

Theorem 6: The set {•, ~} is adequate.
Theorem 7: The set {~, ⊃} is adequate.
The proofs of Theorems 6 and 7 are left to you.

Not all sets of pairs of connectives are adequate.

Theorem 8: The set {⊃, ∨} is inadequate.
Proof 8:

To show that a set of connectives is inadequate, we can show that there is some truth table that can not be constructed using those connectives.
Recall that both 'α ⊃ β' and 'α ∨ β' are true when α and β are both true.
Thus, using these connectives we can never construct a truth table with a false first row.

QED

All of the sets of single connectives that you have seen so far are inadequate.
For example:

Theorem 9: The set {⊃} is inadequate.
Proof 9:
Consider the truth table for conjunction: TFFF.
We want to construct a formula, using ⊃ as the only connective, which yields the same truth table.
Imagine that we have such a formula, and imagine the smallest such formula.
Since, the only way to get an F with ⊃ is with a false consequent, the truth table of the consequent of our formula must either be TFFF or FFFF.
Since we are imagining that our formula is the smallest formula which yields TFFF, the consequent of our formula must be a contradiction.
But, the only way to get a contradiction, using ⊃ alone, is to have one already!
Since we can not construct the contradiction, we can not construct the conjunction.
QED

We will need one more inadequate set, for the proof of Theorem 13.

Theorem 10: The set {~} is inadequate.
Proof 10:
The only possible truth tables with one variable and ~ are TF and FT.
Thus, we can not generate TT or FF.
QED

IV. Adequacy for new connectives

There are sets of single connectives which are adequate.
Consider the Sheffer stroke, '|', which is also called alternative denial, or not-both.

α	|	β
T	F	T
T	T	F
F	T	T
F	T	F

Theorem 11: The set {|} is adequate.
Proof 11:
'$\sim\alpha$' is logically equivalent to '$\alpha \mid \alpha$'.
'$\alpha \bullet \beta$' is logically equivalent to '$(\alpha \mid \beta) \mid (\alpha \mid \beta)$'.
By Theorem 6, {~, •} is adequate.
QED

There is one more adequate single-membered set.
Consider the connective for the Peirce arrow, '↓', also called joint denial, or neither-nor.

α	↓	β
T	F	T
T	F	F
F	F	T
F	T	F

Theorem 12: The set {↓} is adequate.
Proof 12:
'$\sim\alpha$' is equivalent to '$\alpha \downarrow \alpha$'.
'$\alpha \vee \beta$' is equivalent to '$(\alpha \downarrow \beta) \downarrow (\alpha \downarrow \beta)$'.
Theorem 5.
QED

Both | and ↓ were initially explored by C.S. Peirce, though Henry Sheffer gets his name attached to the former for his independent work on it.

V. The limit of adequacy

Lastly, there are no other single, adequate connectives.
Theorem 13: ↓ and | are the only connectives which are adequate by themselves.
Proof 13:
Imagine we had another adequate connective, #.
We know the first rows must be false, by the reasoning in Proof 8.
Similar reasoning fills in the last row.

α	#	β
T	F	T
T		F
F		T
F	T	F

Thus, '~α' is equivalent to 'α # α'.
Now, we need to fill in the other rows.
If the remaining two rows are TT, then we have '|'.
If the remaining two rows are FF, then we have '↓'.
So, the only other possibilities are TF and FT.
FT yields FFTT, which is just '~α'.
TF yields FTFT, which is just '~β'.
By Theorem 10, {~} is inadequate.
QED

VI. Solutions to Exercises A

Only 4 and 8 are not in DNF, though 8 could be quickly put into DNF

VII. For further reading/papers:

Geoffrey Hunter, *Metalogic*. The results above are mostly contained in §21. The references below are mostly found there, as well. His notation is a bit less friendly, but the book is wonderful, and could be the source of lots of papers.

Elliott Mendelson, *Introduction to Mathematical Logic*. Mendelson discusses adequacy in §1.3. His notation is less friendly than Hunter's, but the exercises lead you through some powerful results.

Emil Post, "Introduction to a General Theory of Elementary Propositions", reprinted in van Heijenhoort. The notation is different, but the concepts are not too difficult. It would be interesting to translate into a current notation, and present some of the results.

Several papers from C.S. Peirce initially explored the single adequate connectives. They might be fun to work through. I can give you references.

While there are no other adequate connectives, there are other connectives. You might be able to work up a paper considering some of those.

Also, you could think about why there are only unary and binary connectives.