Philosophy 240: Symbolic Logic Hamilton College
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Class 23: Modal Logic
Fisher 74-84

I. Modal operators

Consider the following sentences:

A. It is not the case that the sun is shining.
B. It is possible that the sun is shining.
C. It is necessary that this sun is shining.

Sentence A contains a sentential operator, negation.
Negation is a function that takes truth values to their opposites.
Sentences B and C contain operators, as well.
These operators are kinds of modal operators, and the logic of these kinds of operators is called modal logic.
Modal logics can be extended into predicate logic, though mostly they are concerned with sentential operators and so most of their interesting properties are available in modal extensions of propositional logic.

The formation rules for modal logic are the same as for propositional logic, with one exception:

Formation rules for propositional modal logic

1. A single capital English letter is a wff.
2. If α is a wff, so is ~α.
3. If α and β are wffs, then so are (α · β), (α ∨ β), (α ⊃ β), and (α ≡ β).
4. If α is a wff, then so is ◊α.
5. These are the only ways to make wffs.

The formal study of modal logics within modern Fregean logic began with C.I. Lewis, in the early twentieth century.
Interest in modal logic became more serious with work from Ruth Barcan Marcus, in the 1940s, and Saul Kripke, in the 1960s.

Modal logic is actually the study of a family of structurally related concepts.
I will introduce a bit of formal modal logic, and stress the primary interpretation (in terms of possibility and necessity) of the modal logic symbols.
The modal logic of possibility and necessity is called alethic logic.
There are several different interpretations of the modal operators.

II. Alethic operators

Possibility is represented using a diamond: ◊
If we take 'S' to stand for 'the sun is shining', then sentence B is regimented as '◊S'.
So, '◊~S' means that it is possible that the sun is not shining.
And, '~◊S' means that it is not possible that the sun is shining.
And, '~◊~S' means that it is not possible that the sun is not shining.
That last sentence is, you may notice, equivalent to sentence C, above.
So, we introduce another operator, '□', for necessity, with the following stipulation:

$\Box\alpha :: \sim\Diamond\sim\alpha$

Similarly:

$\Diamond\alpha :: \sim\Box\sim\alpha$

That is, one or the other modal operator is taken as basic, and the other is introduced by definition.

Studies of alethic operators trace back to Leibniz's interest in possible worlds.
Leibniz thought that we lived in the best of all possible worlds, and that this fact followed from the power and goodness of God.

1. God is omnipotent so he can create the best possible world.
2. God is omni-benevolent, so he wants to create the best possible world.
3. The world exists.

So, this is the best of all possible worlds.
Corollary: All of the evil in this world is necessary.

Voltaire wonderfully lampooned Leibniz's views in *Candide*.
Most philosophers, not only Leibniz, have some interest in possibility and necessity.
For example, much work in the philosophy of mind is motivated by the question of whether it is possible for zombies to exist, whether it is possible for beings physically precisely like me to lack consciousness.

Modal logic gives a way to regiment arguments involving appeals to necessity and possibility.
Kripke's contribution to the modal logic literature was the development of possible world semantics.
Semantics, remember, was what we did with the truth tables.
Semantics is the study of interpretations of formal systems.
Kripke semantics uses the language of possible worlds to interpret sentences of modal logic.
Within the real world, the language of truth tables suffices.

III. Actual World semantics

We've already done semantics for propositional logic using truth tables.
Here is another way to convey the same information, for just negation, conjunction, and the material conditional.
(We've already seen that the biconditional is eliminable.)

$\mathcal{V}(\sim\alpha) = \top$ if $\mathcal{V}(\alpha) = \bot$; otherwise $\mathcal{V}(\sim\alpha) = \bot$
$\mathcal{V}(\alpha \bullet \beta) = \top$ if $\mathcal{V}(\alpha) = \top$ and $\mathcal{V}(\beta) = \top$; otherwise $\mathcal{V}(\alpha \bullet \beta) = \bot$
$\mathcal{V}(\alpha \vee \beta) = \top$ if $\mathcal{V}(\alpha) = \top$ or $\mathcal{V}(\beta) = \top$; otherwise $\mathcal{V}(\alpha \bullet \beta) = \bot$
$\mathcal{V}(\alpha \supset \beta) = \top$ if $\mathcal{V}(\alpha) = \bot$ or $\mathcal{V}(\beta) = \top$; otherwise $\mathcal{V}(\alpha \supset \beta) = \bot$

Consider the following propositions:

P: The penguin is on the TV.
Q: The cat is on the mat.
R: The rat is in the hat.
S: The seal is in the sea.

Suppose that in our world, call it w_1, P, Q, R, and S are all true.
We can easily translate the following claims, and determine their truth values.

Exercises A: Translate each of the following claims into English, and determine their truth values, given the translation key and truth assignments above.

1. $P \supset Q$
2. $P \supset R$
3. $\sim(R \vee S)$
4. $(P \bullet Q) \supset R$
5. $(Q \vee \sim R) \supset \sim P$

Still, it would be possible for there to be another world, call it w_2, in which, say, P and Q are true, but R and S are false.
That is, in w_2, the penguin is on the TV and the cat is on the mat, but the rat is not in the hat and the seal is not in the sea.
To represent the differences between w_1 and w_2, we need to extend actual world semantics to possible world semantics.

IV. Possible World semantics (Leibnizian)

In possible world semantics, valuations at each world are done as above, as a one-place function.
But, we now look at valuations as being a two-place function.
First, we consider a universe, U, which is just a set of worlds: $\{w_1, w_2, w_3,...,w_n\}$
Now, our valuations are indexed to whichever world they are in.

$V(\sim\alpha, w_n) = \top$ if $V(\alpha, w_n) = \bot$; otherwise $V(\sim\alpha, w_n) = \bot$
$V(\alpha \bullet \beta, w_n) = \top$ if $V(\alpha, w_n) = \top$ and $V(\beta, w_n) = \top$; otherwise $V(\alpha \bullet \beta, w_n) = \bot$
$V(\alpha \vee \beta, w_n) = \top$ if $V(\alpha, w_n) = \top$ or $V(\beta, w_n) = \top$; otherwise $V(\alpha \vee \beta) = \bot$
$V(\alpha \supset \beta, w_n) = \top$ if $V(\alpha, w_n) = \bot$ or $V(\beta, w_n) = \top$; otherwise $V(\alpha \supset \beta, w_n) = \bot$

Let's consider a small universe:

$U = \{w_1, w_2, w_3\}$
At w_1, P, Q, R and S are all true.
At w_2, P and Q are true, but R and S are false.
At w_3, P is true, and Q, R, and S are false.

Now, let's return to the sentences from Exercises A.
At w_1, their values remain as above.
We can also evaluate them for w_2 and w_3.

Exercises B: Determine the truth values of each of the following claims at w_2 and w_3.

1. $P \supset Q$
2. $P \supset R$
3. $\sim(R \vee S)$
4. $(P \bullet Q) \supset R$
5. $(Q \vee \sim R) \supset \sim P$

For ease of identification, we can index propositions.
So, we can use 'P_1' to mean that the penguin is on the TV at w_1.

$P_1 \supset P_3$	means that if the penguin is on the TV in w_1, then it is on the TV in w_3
$\sim(Q_2 \bullet Q_3)$	means that it is not the case that the cat is on the mat in both w_2 and w_3

Exercises C: Translate each of the following claims into English, and determine their truth values

1. $P_1 \supset (P_2 \bullet P_3)$
2. $S_1 \bullet S_2 \bullet S_3$
3. $R_1 \vee R_2 \vee R_3$
4. $[(P_1 \vee P_2) \supset (Q_1 \vee Q_2)] \bullet (P_3 \supset \sim Q_3)$

Until now, we have looked at non-modal propositions in other possible worlds.
What about modal claims, like '$\Diamond$S'?
We can translate them, easily, but we need a semantics to determine their truth.
Here is a Leibnizian semantics of the modal operators:

$V(\Box\alpha, w_n) = \top$ if $V(\alpha, w_n) = \top$ for all w_n in U
$V(\Box\alpha, w_n) = \bot$ if $V(\alpha, w_n) = \bot$ for any w_n in U
$V(\Diamond\alpha, w_n) = \top$ if $V(\alpha, w_n) = \top$ for any w_n in U
$V(\Diamond\alpha, w_n) = \bot$ if $V(\alpha, w_n) = \bot$ for all w_n in U

So, '$\Diamond S_1$' is true, since there is at least one world, w_1, in which the seal is in the sea.
Similarly, '$\Box$Q' will be false, since there is at least one world, w_3, in which the cat is not on the mat.

Exercises D: Determine the truth values of each of the following propositions, given the semantics and assignments above.

1. $\Box(P \supset Q)_1$
2. $\Diamond(P \supset Q)_1$
3. $\Box P_1 \supset \Box Q_1$
4. $\Diamond P_1 \supset \Diamond Q_1$
5. $\Diamond[(Q \vee {\sim}R) \supset {\sim}P]_1$
6. $\Diamond P_1 \supset [Q_1 \supset \Box(R \bullet S)_1]$
7. Which of the truth values of the above sentences vary if considered at w_2 or w_3 (i.e. if we replace all the subscripts with '2's or '3's)?

V. Philosophy of modal logic

Let's put aside, for a moment, the technical work.
There are two obvious types of philosophical questions about modal logic.
The first type of question is about the status of possible worlds.
What is the nature of a possible world?
Do they exist?
Are they abstract objects?
Are they other states of this world, or are they independent of us?
These are all metaphysical questions.

The second type of philosophical question about possible worlds is epistemic.
How do we know about possible worlds?
Do we stipulate them?
Do we discover them, or facts about them?
Do we learn about them by looking at our world?
Do we learn about them by pure thought?

The metaphysical questions are linked to the epistemological questions, of course.
If they are real, and independent of the actual world, then how could we possibly know anything about them?

If all we know is about the actual world, then we couldn't have any way of justifying knowledge of alternative possible worlds.

We have some beliefs about possible worlds.
Bush is president.
But, most of us believe that it is possible that he would not have been president.
Kerry could have won the election in 2004.
Talk of possible worlds is just a way of regimenting such beliefs.

David Lewis takes the most extravagant position regarding possible worlds, called modal realism.
The modal realist believes that all possible worlds exist, just as the actual world exists.
Lewis thinks that there is no metaphysical difference in the way that the actual world exists and the way that possible worlds exist.
You might look at his *On the Plurality of Worlds*.
On the opposite end of the spectrum, Quine thinks that all talk of modal logic is faulty, and that there are no possible worlds.
You might look at his "Reference and Modality" and "Three Grades of Modal Involvement".
The first paper is in *From a Logical Point of View*, the second is in *The Ways of Paradox*.
We will return to his position in a moment.

VI. Possible world semantics (Kripkean)

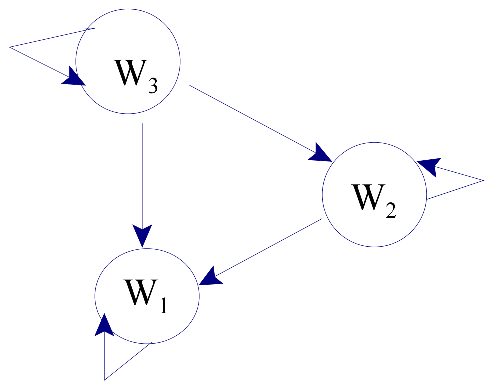
The Leibnizian semantics we have examined suffices to express logical possibility.
But, there are more kinds of possibility than logical possibility.
For example, it is logically possible for a bachelor to be married, for objects to travel faster than the speed of light, and for a square to have five sides.
These claims are logically possible, even though they might be semantically impossible, or physically impossible, or mathematically impossible.
So, a more subtle version of modal logic might be useful.

Consider two possible worlds.
The first world is ours.
The second world is like ours, except that there is some force which moves the planets in perfectly circular orbits.
Now, the law that says that all planets move in elliptical orbits holds in both worlds, since a circle is just a type of ellipse.
So, w_2 obeys all the laws of w_1, but w_1 does not obey all the worlds of w_2.
The modal logician describes this difference in terms of accessibility.
w_2 is accessible from w_1, but w_1 is not accessible from w_2.
Thus, for Kripkean modal logics, in addition to different possible worlds, we need an accessibility relation.
Accessibility relations entail that propositions that are possible at some worlds are not possible at other worlds.

Here is one possible accessibility relation among our worlds w_1, w_2, and w_3.

$$R= \{<w_1, w_1>, <w_2, w_1>,<w_2, w_2>, <w_3, w_1>, <w_3, w_2>, <w_3, w_3>\}$$

Accessibility relations can be given diagrammatically, too.



Now, we extend our Leibnizian semantics into a Kripkean semantics:

$\mathcal{V}(\Box\alpha, w_n) = \top$ if $\mathcal{V}(\alpha, w_m) = \top$ for all w_m in $\mathcal{U}$ such that $<w_n, w_m>$ is in $\mathcal{R}$
$\mathcal{V}(\Box\alpha, w_n) = \bot$ if $\mathcal{V}(\alpha, w_m) = \bot$ for any w_m in $\mathcal{U}$ such that $<w_n, w_m>$ is in $\mathcal{R}$
$\mathcal{V}(\Diamond\alpha, w_n) = \top$ if $\mathcal{V}(\alpha, w_m) = \top$ for any w_m in $\mathcal{U}$ such that $<w_n, w_m>$ is in $\mathcal{R}$
$\mathcal{V}(\Diamond\alpha, w_n) = \bot$ if $\mathcal{V}(\alpha, w_m) = \bot$ for all w_m in $\mathcal{U}$ such that $<w_n, w_m>$ is in $\mathcal{R}$

The introduction of accessibility relations turns modal logic into an enormous field.
We can characterize different kinds of relations.
For example, an accessibility relation might be an equivalence relation: reflexive, symmetrical, and transitive.
A relation is reflexive if every object bears the relation to itself.
In the case above, every world is accessible from itself.
A relation is symmetric if a bearing R to b entails that b bears R to a.
The relation above is not symmetric since w_2 is accessible from w_1, but w_1 is not accessible from w_2.
A relation is transitive if given that a bears R to b and b bears R to c, it follows that a bears R to c.
The case above is transitive.
If the accessibility relation is an equivalence relation, the universe will be Leibnizian.
But, there are much more restrictive kinds of accessibility relations.

Let's go back to our located animals, and see how they fare in a Kripkean universe with the given accessibility relation.

$\mathcal{U} = \{w_1, w_2, w_3\}$
At w_1, P, Q, R and S are all true.
At w_2, P and Q are true, but R and S are false.
At w_3, P is true, and Q, R, and S are false.
$\mathcal{R} = \{<w_1, w_1>, <w_2, w_1>, <w_2, w_2>, <w_3, w_1>, <w_3, w_2>, <w_3, w_3>\}$

Exercises E: Determine the truth values of each of the following formulas.

1. $\Box(P \supset Q)_1$
2. $\Box(P \supset Q)_3$
3. $\Diamond{\sim}(Q \vee R)_1$
4. $\Diamond{\sim}(Q \vee R)_2$
5. $\Diamond{\sim}(Q \vee R)_3$
6. $\Box P_1 \supset \Box Q_1$
7. $\Box P_3 \supset \Box Q_3$
8. $\Diamond[(Q \vee {\sim}R) \supset {\sim}P]_1$
9. $\Diamond[(Q \vee {\sim}R) \supset {\sim}P]_3$

VII. Various other modal systems, and their characteristic axioms

Different modal systems support different modal intuitions, and different interpretations.
These modal systems can be characterized by the modal claims they generate.
For instance, modal system K has the following characteristic axiom.

K: $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$

We can construct an derivation system based on K.
It will be of the form of the axioms systems we examined a couple of weeks ago.
It has the axiom above, as well as the rules of modus ponens, and necessitation:

(Nec) α / $\Box\alpha$

Necessitation looks a little odd, until you recognize that the only things you prove, without premises, are logical truths.
We can also take as a rule that any tautology in propositional logic is a theorem (Thm).
(This is a rule that some of you have asked for, in the past!)
We can derive another rule, called regularity

(Reg) $\alpha \supset \beta$ / $\Box\alpha \supset \Box\beta$

We will show that Reg is valid by deriving '$\Box P \supset \Box Q$' from an arbitrary '$P \supset Q$'

1. $P \supset Q$	/ $\Box P \supset \Box Q$
2. $\Box(P \supset Q)$	1, Nec
3. $\Box(P \supset Q) \supset (\Box P \supset \Box Q)$	Axiom K
4. $\Box P \supset \Box Q$	3, 2, MP

QED

Now, let's do a slightly more complicated proof, of the modal claim: $\Box(P \bullet Q) \supset (\Box P \bullet \Box Q)$

1. $(P \bullet Q) \supset P$	Thm
2. $\Box(P \bullet Q) \supset \Box P$	1, Reg
3. $(P \bullet Q) \supset Q$	Thm
4. $\Box(P \bullet Q) \supset \Box Q$	3, Reg
5. $[\Box(P \bullet Q) \supset \Box P] \supset \{[\Box(P \bullet Q) \supset \Box Q] \supset [\Box(P \bullet Q) \supset (\Box P \bullet \Box Q)]\}$	Thm
6. $[\Box(P \bullet Q) \supset \Box Q] \supset [\Box(P \bullet Q) \supset (\Box P \bullet \Box Q)]$	5, 2, MP
7. $\Box(P \bullet Q) \supset (\Box P \bullet \Box Q)$	6, 4, MP

QED

Note that the theorem used on line 5 has a simple instance: '$(P \supset Q) \supset \{(P \supset R) \supset [P \supset (Q \bullet R)]\}$'

K is a very weak modal logic.
A slightly stronger logic, D, has the characteristic axiom

D: $\Box\alpha \supset \Diamond\alpha$

Every theorem provable in K is provable in D, but D allows for more to be proved.
Thus, it is a stronger logic, and more contentious.
Let's consider the meaning of the characteristic axiom of D.
In alethic interpretation, it means that if a statement is necessary, then it is possible.
This seems reasonable, since necessary statements are all true.
And, true statements are clearly possible.
But, an interesting fact about D is that the following theorem is not provable:

$\Box\alpha \supset \alpha$

Thus, D seems like a poor logic, under alethic interpretation.
There are other interpretations of the modal operators.
One important interpretation is the deontic one.
In deontic logic, '$\Box P$' means that P is obligatory, and '$\Diamond P$' means that P is permissible.
Now, the characteristic axiom of D seems true under this interpretation.
I must be permitted to perform any action that I am obliged to do.
Further, '$\Box\alpha \supset \alpha$' also seems true, since from the fact that an action is obligatory it does not follow that people actually do it.

Another interpretation of the modal logic symbols leads to epistemic logic.
For epistemic logic, we take '$\Box P$' to mean that P is known, and '$\Diamond P$' to mean that P is compatible with things that are known.
Hintikka's epistemic logic takes three axioms:

K: $\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)$
T: $\Box\alpha \supset \alpha$
4: $\Box\alpha \supset \Box\Box\alpha$

Any logic with the T axiom will have a reflexive accessibility relation.
Any logic with the 4 axiom will also have a transitive accessibility relation.

A system with all three of these axioms is called S4.
One can get a slightly stronger logic, by adding a condition of symmetry to the accessibility relation.
Then, we have S5, which takes K, T, 4, and:

B: $\alpha \supset \Box\Diamond\alpha$

There are other interpretations of the modal operators.
A slight tweak of epistemic logic is the logic of belief.
More interestingly, we can generate temporal logics, in which '$\Box$P' means that P is always the case, and '$\Diamond$P' means that P is the case at some point in time.
Temporal logics can be extended to include tense operators, for the future and the past.
I leave explorations of these logics to you.
These are good paper topics.

One last interpretation of the modal operators has become especially important in contemporary philosophy of science and mathematics.
Hartry Field uses the modal operator to represent consistency.
Normally, consistency is a metalogical notion, discussed in a metalanguage.
In advanced logic courses, we reason about the systems we study in introductory logic courses, like this one.
Field argues that the consistency of a set of sentences is actually a logical notion.
Thus, we need to have symbols in the language to represent it.
On Field's view, '$\Diamond$(P • Q • R)' means that P, Q, and R are consistent.

VIII. A criticism of modal logic

Consider the following two claims.

V: The number of planets is greater than seven.
W: Nine is greater than seven.

V and W have the same truth value.
We expect that they will have the same truth value, one can be generated from the other by a simple substitution, given that:

X: The number of planets is nine.

(I will continue to consider Pluto a planet, for the purposes of this example.
We could just change it to eight, if we want.)
Now, make V and W modal claims.

Y: Necessarily, the number of planets is greater than seven.
Z: Necessarily, nine is greater than seven.

Now, Y and Z have different truth values.
But, we can still generate one from the other, with the given substitution.
Quine's criticism of modal logic has been influential; again see "Reference and Modality".