

Class 22: Logical Truths (§7.7)

**I. Sample Derivations of Logical Truths**

We can use either conditional proof or indirect proof to derive logical truths.

If the main connective is a conditional or a biconditional, we generally use conditional proof.

If the main connective is a disjunction or a negation, we generally use indirect proof.

If the main connective is a conjunction, we look to the main connectives of each conjunct to determine the best method of proof.

Using CP:

Show that  $(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$  is a logical truth.

1. $P \supset Q$	ACP
2. $Q \supset R$	ACP
3. $P \supset R$	1, 2, HS
4. $(Q \supset R) \supset (P \supset R)$	2-3, CP
5. $(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$	1-4, CP
QED	

Using IP:

Show that  $(P \supset Q) \vee (\sim Q \supset P)$  is a logical truth.

1. $\sim[(P \supset Q) \vee (\sim Q \supset P)]$	AIP
2. $\sim(P \supset Q) \cdot \sim(\sim Q \supset P)$	1, DM
3. $\sim(P \supset Q)$	2, Simp
4. $\sim(\sim P \vee Q)$	3, Impl
5. $P \cdot \sim Q$	4, DM, DN
6. $\sim(\sim Q \supset P)$	2, Com, Simp
7. $\sim(Q \vee P)$	6, Impl, DN
8. $\sim Q \cdot \sim P$	7, DM
9. $P$	5, Simp
10. $\sim P$	8, Com, Simp
11. $P \cdot \sim P$	9, 10, Conj
12. $(P \supset Q) \vee (\sim Q \supset P)$	1-11, IP
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Sometimes, we might need a logical truth as an intermediate step in a proof:

1. $B \supset [(D \supset D) \supset E]$				
2. $E \supset \{[F \supset (G \supset F)] \supset (H \cdot \sim H)\}$	/ $\sim B$			
3. B			AIP	
4. $(D \supset D) \supset E$			1, 3, MP	Note that 'D $\supset$ D' is derivable using IP or CP
5. $\sim(D \supset D)$			AIP	
6. $\sim(\sim D \vee D)$			5, Impl	
7. $D \cdot \sim D$			6, DM, DN	
8. $D \supset D$			5-7, IP, DN	
9. E			4, 8, MP	
10. $[F \supset (G \supset F)] \supset (H \cdot \sim H)$			2, 9, MP	Note that the antecedent here is another logical truth
11. F			ACP	
12. $F \vee \sim G$			11, Add	
13. $\sim G \vee F$			12, Com	
14. $G \supset F$			13, Impl	
15. $F \supset (G \supset F)$			11-14, CP	
16. $H \cdot \sim H$			10, 15, MP	
17. $\sim B$			3-16, IP	
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III. **Exercises.** Derive each of the following logical truths, using CP or IP.

1.  $[(A \supset B) \cdot A] \supset B$
2.  $(P \vee P) \supset P$
3.  $(A \supset B) \supset [A \supset (A \cdot B)]$
4.  $(A \supset B) \vee (\sim A \supset C)$
5.  $(P \supset Q) \supset \{(P \supset R) \supset [P \supset (Q \cdot R)]\}$

IV. Practice for the next exam. Some proofs for review:

$$1. \quad \begin{array}{l} 1. G \supset H \\ 2. G \supset I \end{array} \quad / G \supset (H \cdot I)$$

$$2. \quad \begin{array}{l} 1. J \equiv K \\ 2. K \equiv L \end{array} \quad / J \equiv L$$

$$3. \quad 1. M \quad / N \supset \{O \supset \{P \supset [(M \cdot N) \cdot (O \cdot P)]\}\}$$

$$4. \quad / R \supset [(R \supset S) \supset S]$$

$$5. \quad \begin{array}{l} 1. D \supset E \\ 2. E \supset (F \cdot G) \\ 3. \sim F \vee \sim G \end{array} \quad / \sim D$$

$$6. \quad \begin{array}{l} 1. (P \vee Q) \vee (\sim R \vee S) \\ 2. \sim Q \cdot (\sim S \cdot \sim P) \end{array} \quad / \sim R$$

$$7. \quad / A \vee [(\sim A \vee B) \cdot (\sim A \vee C)]$$

$$8. \quad \begin{array}{l} 1. (A \vee B) \supset \sim(F \cdot D) \\ 2. \sim(A \cdot \sim D) \\ 3. \sim F \supset \sim(C \cdot D) \\ 4. C \vee A \end{array} \quad / A \equiv \sim C$$

$$9. \quad \begin{array}{l} 1. (A \cdot B) \supset C \\ 2. (F \cdot D) \supset E \\ 3. A \cdot D \\ 4. B \vee F \end{array} \quad / C \vee E$$

$$10. \quad \begin{array}{l} 1. \sim F \supset (G \cdot H) \\ 2. F \supset (I \vee J) \\ 3. \sim[G \cdot (K \cdot \sim J)] \\ 4. K \equiv (L \cdot \sim J) \end{array} \quad / K \supset I$$

V. Sample Solutions to the Exercises to Review  
**Solutions may vary!**

1.     1.  $G \supset H$   
        2.  $G \supset I$                      / $G \supset (H \cdot I)$   
           | 3.  $G$                      ACP  
           | 4.  $H$                      1, 3, MP  
           | 5.  $I$                      2, 3, MP  
           | 6.  $H \cdot I$                4, 5, Conj  
        7.  $G \supset (H \cdot I)$            3-6, CP

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2.     1.  $J \equiv K$   
        2.  $K \equiv L$                      / $J \equiv L$   
        3.  $(J \supset K) \cdot (K \supset J)$        1, Equiv  
        4.  $(K \supset L) \cdot (L \supset K)$        2, Equiv  
        5.  $J \supset K$                      3, Simp  
        6.  $K \supset L$                      4, Simp  
        7.  $J \supset L$                      5, 6, HS  
        8.  $L \supset K$                      4, Com, Simp  
        9.  $K \supset J$                      3, Com, Simp  
        10.  $L \supset J$                     8, 9, HS  
        11.  $(J \supset L) \cdot (L \supset J)$      7, 10, Conj  
        12.  $J \equiv L$                    11, Equiv

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3.     1.  $M$      / $N \supset \{O \supset \{P \supset [(M \cdot N) \cdot (O \cdot P)]\}\}$   
        | 2.  $N$    ACP  
        |     | 3.  $O$                                        ACP  
        |     |     | 4.  $P$                                    ACP  
        |     |     | 5.  $M \cdot N$                            1, 2, Conj  
        |     |     | 6.  $O \cdot P$                            3, 4, Conj  
        |     |     | 7.  $(M \cdot N) \cdot (O \cdot P)$        5, 6, Conj  
        |     |     | 8.  $P \supset [(M \cdot N) \cdot (O \cdot P)]$      4-7, CP  
        |     | 9.  $O \supset \{P \supset [(M \cdot N) \cdot (O \cdot P)]\}$    3-8, CP  
        10.  $N \supset \{O \supset \{P \supset [(M \cdot N) \cdot (O \cdot P)]\}\}$  2-9, CP

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4.     / $R \supset [(R \supset S) \supset S]$   
        | 1.  $R$                                      ACP  
        |     | 2.  $R \supset S$                            ACP  
        |     | 3.  $S$                                2, 1, MP  
        | 4.  $(R \supset S) \supset S$  2-3, CP  
        5.  $R \supset [(R \supset S) \supset S]$            1-4, CP

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- 5.
- |                            |              |
|----------------------------|--------------|
| 1. $D \supset E$           |              |
| 2. $E \supset (F \cdot G)$ |              |
| 3. $\sim F \vee \sim G$    | $/\sim D$    |
| 4. $D$                     | AIP          |
| 5. $E$                     | 1, 4, MP     |
| 6. $F \cdot G$             | 2, 5, MP     |
| 7. $F$                     | 6, Simp      |
| 8. $\sim G$                | 3, 7, DN, DS |
| 9. $G$                     | 6, Com, Simp |
| 10. $G \cdot \sim G$       | 9, 8, Conj   |
| 11. $\sim D$               | 4-10, IP     |

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- 6.
- |   |                |
|---|----------------|
| 1. $(P \vee Q) \vee (\sim R \vee S)$    |                |
| 2. $\sim Q \cdot (\sim S \cdot \sim P)$ | $/\sim R$      |
| 3. $\sim Q$                             | 2, Simp        |
| 4. $(Q \vee P) \vee (\sim R \vee S)$    | 1, Com         |
| 5. $Q \vee [P \vee (\sim R \vee S)]$    | 4, Assoc       |
| 6. $P \vee (\sim R \vee S)$             | 5, 3, DS       |
| 7. $\sim S \cdot \sim P$                | 2, Com, Simp   |
| 8. $\sim P$                             | 7, Com, Simp   |
| 9. $\sim R \vee S$                      | 6, 8, DS       |
| 10. $\sim S$                            | 7, Simp        |
| 11. $\sim R$                            | 9, 10, Com, DS |

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7.  $/A \vee [(\sim A \vee B) \cdot (\sim A \vee C)]$
- |  |              |
|--|--------------|
| 1. $\sim \{A \vee [(\sim A \vee B) \cdot (\sim A \vee C)]\}$   | AIP          |
| 2. $\sim A \cdot \sim [(\sim A \vee B) \cdot (\sim A \vee C)]$ | 1, DM        |
| 3. $\sim A$  | 2, Simp      |
| 4. $\sim [(\sim A \vee B) \cdot (\sim A \vee C)]$              | 2, Com, Simp |
| 5. $\sim [\sim A \vee (B \cdot C)]$                            | 4, Dist      |
| 6. $A \cdot \sim (B \cdot C)$                                  | 5, DM, DN    |
| 7. $A$   | 6, Simp      |
| 8. $A \cdot \sim A$  | 7, 3, Conj   |
| 9. $A \vee [(\sim A \vee B) \cdot (\sim A \vee C)]$            | 1-8, IP, DN  |

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|----|---|---------------------|
| 8. | 1. $(A \vee B) \supset \sim(F \cdot D)$           |                     |
|    | 2. $\sim(A \cdot \sim D)$                         |                     |
|    | 3. $\sim F \supset \sim(C \cdot D)$               |                     |
|    | 4. $C \vee A$                                     | / $A \equiv \sim C$ |
|    | 5. A  | ACP                 |
|    | 6. $A \vee B$                                     | 5, Add              |
|    | 7. $\sim(F \cdot D)$                              | 6, 1, 6, MP         |
|    | 8. $\sim F \vee \sim D$                           | 7, DM               |
|    | 9. $\sim A \vee \sim \sim D$                      | 2, DM               |
|    | 10. $\sim \sim D$                                 | 9, 5, DM, DS        |
|    | 11. $\sim F$                                      | 8, 10, Com, DS      |
|    | 12. $\sim(C \cdot D)$                             | 3, 11, MP           |
|    | 13. $\sim C \vee \sim D$                          | 12, DM              |
|    | 14. $\sim C$                                      | 13, 10, Com, DS     |
|    | 15. $A \supset \sim C$                            | 5-14, CP            |
|    | 16. $\sim C$                                      | ACP                 |
|    | 17. A   | 4, 16, DS           |
|    | 18. $\sim C \supset A$                            | 16, 17, CP          |
|    | 19. $(A \supset \sim C) \cdot (\sim C \supset A)$ | 15, 18, Conj        |
|    | 20. $A \equiv \sim C$                             | 19, Equiv           |

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|----|----------------------------|---------------|
| 9. | 1. $(A \cdot B) \supset C$ |               |
|    | 2. $(F \cdot D) \supset E$ |               |
|    | 3. $A \cdot D$             |               |
|    | 4. $B \vee F$              | / $C \vee E$  |
|    | 5. $\sim(C \vee E)$        | AIP           |
|    | 6. $\sim C \cdot \sim E$   | 5, DM         |
|    | 7. $\sim C$                | 6, Simp       |
|    | 8. $\sim(A \cdot B)$       | 1, 7, MT      |
|    | 9. $\sim A \vee \sim B$    | 8, DM         |
|    | 10. A                      | 3, Simp       |
|    | 11. $\sim B$               | 9, 10, DN, DS |
|    | 12. F                      | 4, 11, DS     |
|    | 13. D                      | 3, Com, Simp  |
|    | 14. $F \cdot D$            | 12, 13, Conj  |
|    | 15. E                      | 2, 14, MP     |
|    | 16. $\sim E$               | 6, Com, Simp  |
|    | 17. $E \cdot \sim E$       | 15, 16, Conj  |
|    | 18. $C \vee E$             | 5-17, IP, DN  |

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|-----|--|-----------------|
| 10. | 1. $\sim F \supset (G \cdot H)$                                      |                 |
|     | 2. $F \supset (I \vee J)$  |                 |
|     | 3. $\sim [G \cdot (K \cdot \sim J)]$                                 |                 |
|     | 4. $K \equiv (L \cdot \sim J)$                                       | / $K \supset I$ |
|     | 5. $K$   | ACP             |
|     | 6. $[K \supset (L \cdot \sim J)] \cdot [(L \cdot \sim J) \supset K]$ | 4, Equiv        |
|     | 7. $K \supset (L \cdot \sim J)$                                      | 6, Simp         |
|     | 8. $L \cdot \sim J$  | 7, 5, MP        |
|     | 9. $\sim J$  | 8, Com, Simp    |
|     | 10. $K \cdot \sim J$   | 5, 9, Conj      |
|     | 11. $\sim G \vee \sim (K \cdot \sim J)$                              | 3, DM           |
|     | 12. $\sim (K \cdot \sim J) \vee \sim G$                              | 11, Com         |
|     | 13. $\sim G$   | 12, 10, DN, DS  |
|     | 14. $\sim I$   | AIP             |
|     | 15. $\sim I \cdot \sim J$  | 14, 9, Conj     |
|     | 16. $\sim (I \vee J)$  | 15, DM          |
|     | 17. $\sim F$   | 2, 16, MT       |
|     | 18. $G \cdot H$  | 1, 17, MP       |
|     | 19. $G$  | 18, Simp        |
|     | 20. $G \cdot \sim G$   | 19, 13, Conj    |
|     | 21. $I$  | 14-20, IP       |
|     | 22. $K \supset I$  | 5-21, CP        |

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