Class 21: Indirect Proof (§7.6)

I. **Indirect Proof: Another Method for Derivation**

Consider the following two proofs:

1.
1.	1. A · ~A	/ B
	2. A	1, Simp
	3. A ∨ B	2, Add
	4. ~A	1, Com, Simp
	5. B	3, 4, DS

QED

2.
2.	1. B ⊃ (P · ~P)	/ ~B
	| 2. B	
	| 3. P · ~P	
	| 4. P	
	| 5. P ∨ ~B	
	| 6. ~P	
	| 7. ~B	
	8. B ⊃ ~B	
	9. ~B ∨ ~B	
	10. ~B	

QED

The moral of the first is that anything follows from a contradiction.
The moral of the second is that if a statement entails a contradiction, then its negation is true.
Indirect proof is based on these two morals.
Indirect proof is also called reductio ad absurdum, or just reductio.
Assume your desired conclusion is false, and try to get a contradiction.
If you get it, then you know the opposite of the assumption is true.

Procedure for Indirect Proof, (IP)

1. Indent, assuming the opposite of what you want to conclude (one more or one fewer '~').
2. Derive a contradiction, using any letter.
3. Discharge the negation (not the opposite!) of your assumption.

A contradiction is any statement of the form: α • ~α
The following wffs are all contradictions:

P •~ P
~~P • ~~~P
~(P∨ ~Q) • ~~(P ∨ ~Q)

Sample Derivation:

1. A ⊃ B
2. A ⊃ ~B /~A
   | 3. A AIP Let's see what happens if the opposite of the conclusion is true.
   | 4. B 1, 3, MP
   | 5. ~B 2, 3, MP
   | 6. B · ~B 4, 5, Conj This is impossible - a contradiction.
7. ~A 3-6, IP So ~~A must be false, and so ~A is true.

QED

The method of indirect proof is especially useful for proving disjunctions as well as simple statements and negations.

II. **More sample derivations**

Plain indirect proof:

1. F ⊃ ~D
2. D
3. (D · ~E) ⊃ F / E
   | 4. ~E AIP
   | 5. D · ~E 2, 4, Conj
   | 6. F 3, 5, MP
   | 7. ~D 1, 6, MP
   | 8. D · ~D 2, 7, Conj
9. ~~E 4-8, CP
10. E 9, DN

QED

Indirect proof with conditional proof:

1. E ⊃ (A · D)
2. B ⊃ E / (E ∨ B) ⊃ A
   | 3. E ∨ B ACP
   | | 4. ~A AIP
   | | 5. ~A ∨ ~D 4, Add
   | | 6. ~(A · D) 5, DM
   | | 7. ~E 1, 6, MT
   | | 8. B 3, 7, DS
   | | 9. ~B 2, 7, MT
   | | 10. B · ~B 8, 9, Conj
   | 11. ~~A 4-10, IP
   | 12. A 11, DN
12. (E ∨ B) ⊃ A 3-12, CP

QED

This last method has the form of many (almost all?) proofs in mathematics.
First, one states one's assumptions, one's specific axioms.
Then one assumes one's conclusion is false, and derives a contradiction.

III. **Proving logical truths**

The methods of conditional and indirect proof are easily adapted to proving logical truths.
If a statement is a logical truth, it does not depend on any premises.
It follows from any premises, and even from no premises at all.
With CP and IP, then, you can prove statements without any assumptions.

To prove that '~[(X ≡ Y) · ~(X ∨ ~Y)]' is a logical truth, we start with an assumption.

| 1. (X ≡ Y) · ~(X ∨ ~Y)	AIP
| 2. X ≡ Y	1, Simp
| 3. (X ⊃ Y) · (Y ⊃ X)	2, Equiv
| 4. ~(X ∨ ~Y)	1, Com, Simp
| 5. ~X · Y	4, DM DN
| 6. Y ⊃ X	3, Com, Simp
| 7. ~X	5, Simp
| 8. ~Y	6, 7, MT
| 9. Y	5, Com, Simp
| 10. Y · ~Y	9, 8, Conj
11. ~[(X ≡ Y) · ~(X ∨ ~Y)]	1-10, IP

QED

CP is also useful to prove arguments without premises.

Example: Prove that '[P ⊃ (Q ⊃ R)] ⊃ [(P ⊃ Q) ⊃ (P ⊃ R)]' is a tautology.

| 1. P ⊃ (Q ⊃ R)	ACP	(to prove (P ⊃ Q) ⊃ (P ⊃ R))
| | 2. P ⊃ Q	ACP	(to prove (P ⊃ R))
| | | 3. P	ACP	(to prove R)
| | | 4. Q ⊃ R	1, 3, MP	
| | | 5. Q	2, 3, MP	
| | | 6. R	4, 5, MP	
| | 7. P ⊃ R	3-6 CP	
| 8. (P ⊃ Q) ⊃ (P ⊃ R)	2-7, CP	
9. [P ⊃ (Q ⊃ R)] ⊃ [(P ⊃ Q) ⊃ (P ⊃ R)]	1-8, CP	

QED

Note that the last line of both proofs is further un-indented than the first line, since the first line is indented.
The conclusions, here, are not part of the proofs, and have no line number until the end.

IV. **Exercises**. Derive the conclusions of the following arguments using the 18 rules, and either CP or IP.

1. 1. A ⊃ B
   2. ~A ∨ ~B / ~A

2. 1. F ⊃ (~E ∨ D)
   2. F ⊃ ~D / F ⊃ ~E

3. 1. ~J ⊃ (G · H)
   2. G ⊃ I
   3. H ⊃ ~I / J

4. 1. S ⊃ (T ∨ U)
   2. W ⊃ ~U / S ⊃ ~(W · ~T)

5. 1. (L ⊃ M) · (N ⊃ O)
   2. (M ∨ O) ⊃ P
   3. ~P / ~(L ∨ N)

6 Prove that '(A ⊃ B) ∨ (B ⊃ A)' is a logical truth.

7. Prove that '(P ⊃ Q) ⊃ [(P · R) ⊃ (Q · R)]' is a logical truth.

8. Prove that '(P · Q) ⊃ [(P ∨ R) · (Q ∨ R)]' is a logical truth.

Solutions may vary.