Class 17: Rules of Replacement, II (§7.4)

I. **The Last Five Rules of Replacement**

Transposition (Trans)

$\alpha \supset \beta :: \sim\beta \supset \sim\alpha$

You may switch the antecedent and consequent of a conditional statement, as long as you negate (or un-negate) both.
Often used with HS.
A statement and its transposition are traditionally called contrapositives of each other.

Sample Derivation:

1. A ⊃ B
2. D ⊃ ~B / A ⊃ ~D
3. ~~B ⊃ ~D 2, Trans
4. A ⊃ ~D 1, 3, DN, HS

QED

Transposition can be tricky when only one term is negated:

A ⊃ ~B becomes, by Trans:
~~B ⊃ ~A which becomes, by DN
B ⊃ ~A

Equivalently, but doing the double negation first:

A ⊃ ~B becomes, by DN:
~~A ⊃ ~B becomes, by Trans:
B ⊃ ~A

Either way, you can include the DN on the line with Trans.

Material Implication (Impl)

$\alpha \supset \beta :: \sim\alpha \vee \beta$

Implication allows you to change a statement from a disjunction to a conditional, or vice versa.
It is often easier to work with disjunctions.
You can use DM to get conjunctions.
You may be able to use distribution, which doesn't apply to conditionals.
On the other hand, sometimes, you just want to work with conditionals.
You can use HS and MP.
Proofs are overdetermined by our system - there are many ways to do them.

Sample Derivation:

1. $G \supset \sim E$
2. $E \vee F$ / $G \supset F$
3. $\sim\sim E \vee F$ 2, DN
4. $\sim E \supset F$ 3, Impl
5. $G \supset F$ 1, 4, HS

QED

Material Equivalence (Equiv)

$\alpha \equiv \beta :: (\alpha \supset \beta) \cdot (\beta \supset \alpha)$

$\alpha \equiv \beta :: (\alpha \cdot \beta) \vee (\sim\alpha \cdot \sim\beta)$

Equiv is almost the only thing you can do with a biconditional.
There are wo distinct versions.
If you have a biconditional in your premises, you can unpack it in either way.
If you need one in your conclusion, you can get the pieces and then use this rule.
This is easier with the first definition.
Just get $\alpha \supset \beta$
Then get $\beta \supset \alpha$
Then use Conj.

Sample Derivation

1. $\sim[(K \supset \sim H) \cdot (\sim H \supset K)]$
2. $(I \cdot J) \supset (K \equiv \sim H)$ / $\sim(I \cdot J)$
3. $\sim(K \equiv \sim H)$ 1, Equiv
4. $\sim(I \cdot J)$ 2, 3, MT

QED

Exportation (Exp)

$\alpha \supset (\beta \supset \gamma) :: (\alpha \cdot \beta) \supset \gamma$

You can sometimes get to MP or MT using Exportation.

Sample Derivation:

1. $L \supset (M \supset N)$
2. $\sim N$ / $\sim L \vee \sim M$
3. $(L \cdot M) \supset N$ 1, Exp
4. $\sim(L \cdot M)$ 3, 2, MT
5. $\sim L \vee \sim M$ 4, DM

QED

Tautology (Taut)

α :: α · α
α :: α ∨ α

Tautology eliminates redundancy.

Sample Derivation:

1. O ⊃ ~O	/ ~O
2. ~O ∨ ~O	1, Impl
3. ~O	2, Taut
QED	

II. **Some more potentially helpful examples**

Some of these derivations may be useful as elements of other, longer proofs.
Others contain useful tricks which may come in handy in other proofs.

1.

1. ~A	/ A ⊃ B
2. ~A ∨ B	1, Add
3. A ⊃ B	2, Impl
QED	

2.

1. E	/ F ⊃ E
2. ~F ∨ E	1, Add, Com
3. F ⊃ E	2, Impl
QED	

3.

1. G ⊃ (H ⊃ I)	/ H ⊃ (G ⊃ I)
2. (G · H) ⊃ I	1, Exp
3. (H · G) ⊃ I	2, Com
4. H ⊃ (G ⊃ I)	3, Exp
QED	

4.

1. O ⊃ (P · Q)	/ O ⊃ P	
2. ~O ∨ (P · Q)		1, Impl
3. (~O ∨ P) · (~O ∨ Q)		2, Dist
4. ~O ∨ P		3, Simp
5. O ⊃ P		4, Impl
QED		

5.

1. (R ∨ S) ⊃ T	/ R ⊃ T
2. ~(R ∨ S) ∨ T	1, Impl
3. (~R · ~S) ∨ T	2, DM
4. T ∨ (~R · ~S)	3, Com
5. (T ∨ ~R) · (T ∨ ~S)	4, Dist
6. ~R ∨ T	5, Simp, Com
7. R ⊃ T	6, Impl
QED	

6.
1. W ⊃ X
2. Y ⊃ X   / (W ∨ Y) ⊃ X
3. (W ⊃ X) · (Y ⊃ X)   1, 2, Conj
4. (~W ∨ X) · (~Y ∨ X)   3, Impl, Impl
5. (X ∨ ~W) · (X ∨ ~Y)   4, Com, Com
6. X ∨ (~W · ~Y)   5, Dist
7. (~W · ~Y) ∨ X   6, Com
8. ~(W ∨ Y) ∨ X   7, DM
9. (W ∨ Y) ⊃ X   8, Impl
QED

7.
1. (J ∨ K) ⊃ (L · M)
2. ~J ⊃ (N ⊃ ~N)
3. ~L   / ~N
4. ~L ∨ ~M   3, Add
5. ~(L · M)   4, DM
6. ~(J ∨ K)   1, 5, MT
7. ~J ·~K   6, DM
8. ~J   7, Simp
9. N ⊃ ~N   2, 8, MP
10. ~N ∨ ~N   9, Impl
11. ~N   10, Taut
QED

III. **Exercises**. Derive the conclusions of each of the following arguments using the Rules of Inference and Replacement.

1.
1. (O · P) ⊃ Q
2. O   / P ⊃ Q

2.
1. R ⊃ (S · ~T)   / ~R ∨ ~T

3.
1. U ≡ W
2. W   / U

4.
1. (H ∨ I) ⊃ [J · (K · L)]
2. I   / J · K

5.
1. (L · M) ⊃ N
2. (L ⊃ N) ⊃ O   / M ⊃ O

6.
1. A · (B ∨ F)
2. A ⊃ [B ⊃ (D · E)]
3. (A · F) ⊃ ~(D ∨ E)   / D ≡ E

Solutions may vary.

IV. **Three challenging derivations**. Try them.

1.
    1. $A \supset B$
    2. $B \supset D$
    3. $D \supset A$
    4. $A \supset {\sim}D$ / ${\sim}A \cdot {\sim}D$

2.
    1. $(I \cdot E) \supset {\sim}F$
    2. $F \vee (G \cdot H)$
    3. $I \equiv E$ / $I \supset G$

3.
    1. $(J \supset J) \supset (K \supset K)$
    2. $(K \supset L) \supset (J \supset J)$ / $K \supset K$

V. **Appendix: Proofs of the Logical Equivalence of the Last Five Rules of Replacement**

Transposition: α ⊃ β :: ~β ⊃ ~α

α	⊃	β
T	**T**	T
T	**F**	F
F	**T**	T
F	**T**	F

~	β	⊃	~	α
F	T	**T**	F	T
T	F	**F**	F	T
F	T	**T**	T	F
T	F	**T**	T	F

Material Implication: α ⊃ β :: ~α ∨ β

α	⊃	β
T	**T**	T
T	**F**	F
F	**T**	T
F	**T**	F

~	α	∨	β
F	T	**T**	T
F	T	**F**	F
T	F	**T**	T
T	F	**T**	F

Material Equivalence: α ≡ β :: (α ⊃ β) · (β ⊃ α)

α	≡	β
T	**T**	T
T	**F**	F
F	**F**	T
F	**T**	F

(α	⊃	β)	·	(β	⊃	α)
T	T	T	**T**	T	T	T
T	F	F	**F**	F	T	T
F	T	T	**F**	T	F	F
F	T	F	**T**	F	T	F

Material Equivalence: α ≡ β :: (α · β) ∨ (~α · ~β)

α	≡	β
T	**T**	T
T	**F**	F
F	**F**	T
F	**T**	F

(α	·	β)	∨	(~	α	·	~	β)
T	T	T	**T**	F	T	F	F	T
T	F	F	**F**	F	T	F	T	F
F	F	T	**F**	T	F	F	F	T
F	F	F	**T**	T	F	T	T	F

Exportation: (α · β) ⊃ γ :: α ⊃ (β ⊃ γ)

(α	·	β)	⊃	γ
T	T	T	**T**	T
T	T	T	**F**	F
T	F	F	**T**	T
T	F	F	**T**	F
F	F	T	**T**	T
F	F	T	**T**	F
F	F	F	**T**	T
F	F	F	**T**	F

α	⊃	(β	⊃	γ)
T	**T**	T	T	T
T	**F**	T	F	F
T	**T**	F	T	T
T	**T**	F	T	F
F	**T**	T	T	T
F	**T**	T	F	F
F	**T**	F	T	T
F	**T**	F	T	F

Tautology: α :: α ∨ α

α	∨	α
T	**T**	T
F	**F**	F

Tautology: α :: α · α

α	•	α
T	**T**	T
F	**F**	F