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Class 16: Propositions and Logical Truths
Fisher, pp 46-58

I. Logical Truths

Compare the following three sentences:

A. If it is raining, then I will be unhappy.
B. If it is raining, then I will get wet.
C. If it is raining, then it is raining.

A and B are contingent sentences, each representable as ‘P ⊃ Q’
B is a little bit more compelling, but it is still possible for both sentences to be false.
C, on the other hand, can never be false, as long as we hold the meanings of the terms constant.
It is of the form ‘P ⊃ P’, and it is a logical truth, or a law of logic.

We are going to look at two related topics concerning the laws of logic.
First, we are going to look at what the laws of logic hold of.
We are going to talk a bit about the metaphysical properties of the statements of logic.

Second, the proofs that we have been doing have all taken some contingent assumptions as premises.
Some systems, the ones that interest philosophers most, focus only on logical truths.
Since logical truths are tautologies, they do not depend on any premises.
So, we should be able to prove them without any premises.
But, we do not have that ability, yet.
We will look at two techniques for proving logical truths in Hurley’s system in in two weeks.
Today, I will show you a different kind of system, built for proving the logical truths.

Logical truths are the theorems of our logical theory, just as certain geometric statements are theorems of Euclidean geometry.
A theory is just a set of sentences.
Theorems are the sentences that characterize a theory.
Some theories are axiomatic, and we will look a bit at axiomatic theories, in a moment.
Our theory has no axioms.
All the laws are logical truths, truths which, like C, are necessary.

II. Types, tokens and propositions

Consider the law of the excluded middle: $\alpha \lor {\sim}\alpha$.
We know that ‘$\alpha \lor {\sim}\alpha$’ is a sentence in the meta-language.
It is a schema, a rule for generating particular sentences.
That law tells us that any substitution instance of itself is a tautology.
But, what are the substitution instances?
That is, what are we putting in the place of the ‘α’s?

We'll start with a distinction between types and tokens.
Consider: 'The cat dances. The cat dances.'
There are two sentence tokens, but only one sentence type.
Similarly, in 'Mississippi', there are four letter types, but eleven letter tokens.

The laws of logic do not hold merely of sentence tokens.
Consider disjunctive syllogism:

$\alpha \vee \beta$
$\sim\alpha \quad / \beta$

We have to substitute the same thing for α in both instances, and the same thing for β in both instances.
We can't substitute the same sentence token.
If one token is in the first place, a different token would have to be put in the second place.
So, the laws of logic can't be about sentence tokens.

Perhaps what we want are sentence types.
Consider:

The cat either dances or sings.
She doesn't dance.
Therefore, she sings.

This seems to be an instance of disjunctive syllogism.
But there are different sentence types replacing α and different sentence types replacing β.
The first premise isn't even of the form '$\alpha \vee \beta$', on the surface.
We can recast it in that form, as, 'The cat dances or the cat sings'.
Then 'the cat sings' is what replaces Q in the first premise.
Then, we substitute 'she sings' in the conclusion, which is still a different sentence type from 'the cat sings'.
Similar remarks hold for what replaces β.
While we can recast the argument so that we have exactly the same sentence types in precisely the right positions, we need not do so in order to conclude that the argument above is a version of disjunctive syllogism.
So, it looks like disjunctive syllogism doesn't hold of sentence types, either.
So, the type-token distinction does not seem to help us, but what other option do we have?

Consider 'El gato baila'.
This sentence expresses the same thing as 'the cat dances'.
What the two have in common are their meanings.
These are also called propositions.
Propositions are the meanings of sentence types.
Notice that we do not see or hear propositions.
We encounter sentence tokens, not types.
We infer the type from the presence of the token.
Tokens are concrete, and types are abstract.
Thus, propositions are abstract objects.

Notice that we can both think of the same proposition.
But propositions are not thoughts.
My thoughts are in my mind, your thoughts are in your mind.
Our thoughts may share the same content, but these thoughts, like inscriptions, are tokens.
In fact, some of our mental states are called propositional attitudes.
The propositional attitudes include beliefs and desires.
If I believe that it is snowing, or I desire that it snow, my mental state can be described as a relation between me and a proposition, that it is snowing.
Propositions are the objects of thought, but they are independent of any particular mind.
(Though, Frege called propositions thoughts.)

Propositions may be indicated by ‘that’.
The sentences ‘the cat dances’ and ‘el gato baila’ both express the proposition *that the cat dances*.
But, propositions are also language-independent.
They may be expressed by sentences of language, but they technically belong to no language.
We might call them states of affairs, as well.
What a sentence means is not in a language at all, just as the object to which a term refers is not a linguistic object.

Compare our knowledge of propositions with our knowledge of ‘2’ and ‘2+2=4’.
Mathematical objects, like propositions, are abstract.
‘2’ is the name of a number, but is not the number itself, just as I am not ‘Russell’.
(Of course I am Russell; remember that the ‘’ marks turn our attention to the term inside the ‘’s from that to which it refers or that which it means.
So, a cat is an animal, but ‘cat’ has three letters. And ‘cat’ is a word, not an animal.)
We learn about mathematics in part by using names of mathematical objects, and diagrams.
Similarly, we learn of propositions from our interactions with sentence tokens.
The sentence tokens we use in our proofs are names of propositions.

Here is a last example of why we deal with propositions, and not sentence types.
Consider, ‘Visiting relatives can be annoying’.
This sentence token corresponds to a single sentence type, but it is ambiguous between two propositions.
If we try to substitute a sentence type in a rule of inference, like disjunctive syllogism, we are liable to generate false inferences because of the ambiguity.
Technically, we only substitute propositions, though, of course we only write down tokens representing, or expressing, those propositions.

The view I have just presented might be called the traditional view, or the standard view.
But, it has come under attack for the last century or so.
Quine, who is much smarter about both logic and philosophy than I am, thinks these arguments are all fallacious, and that there are no such things as propositions.
In fact, he calls propositions creatures of darkness.
He argues that belief in meanings is a myth, the myth of the museum.
See his *Philosophy of Logic*; the more ambitions could look at his *Word and Object*.
Fisher agrees with Quine’s view, as do many others.
Later Wittgenstein is also skeptical about meanings.
This skepticism is a central theme of his *Philosophical Investigations*.
But, as Descartes said in a completely different context, “Some people have such a confused conception

of everything and cling so tenaciously to their preconceived opinions that rather than change them they will deny [of themselves] what they cannot fail to experience [within themselves] all the time" (from *Sixth Objections*, AT426-7)..

III. Logical truths

Let's go back to the law of the excluded middle.
Such laws and the rules of inference are schemas written in the meta-language.
They are not propositions themselves, but they tell you how to form certain propositions, or how to infer some propositions from others.
Any substitution instance of the metalinguistic rule is a tautology.
Substitution instances are propositions, but the inscriptions we use are names of those propositions.

The propositions we form from logical laws are sometimes called logical truths.
(Though, sometimes we will call the incomplete sentence of the metalanguage a logical truth, as well.)
'It is raining, now' requires, for it to be true, some justification outside of logic.
It must actually be raining in order for that sentence to be true.
Logical truths require no justification outside of logic.
They can be derived regardless of any assumptions, or premises.
In fact, they can be derived with no premises at all, as we will see, after Fall Break.

The logical truths are the theorems of the logical theory we are using.
In addition to the law of the excluded middle, we have seen the law of non-contradiction (or sometimes, ironically, the law of contradiction): $\sim(\alpha \bullet \sim\alpha)$.
There are infinitely many theorems.
Here are two other examples of schemas for producing logical truths of propositional logic:

$$\alpha \supset (\beta \supset \alpha)$$
$$[\alpha \supset (\beta \supset \gamma)] \supset [(\alpha \supset \beta) \supset (\alpha \supset \gamma)]$$

IV. Axiom systems

Some systems of logic have a rule by which you can enter laws of logic into any line in a proof.
Then we need an independent method for determining that a statement is a theorem.
Truth tables suffice for this.
Some systems take a few logical truths as axioms, and reduce the rules of inference, even to just one rule, normally MP.

For example, consider the following system, which takes three axioms, and just the rules of modus ponens, and substitution:

Ax_1: $\alpha \supset (\beta \supset \alpha)$
Ax_2: $[\alpha \supset (\beta \supset \gamma)] \supset [(\alpha \supset \beta) \supset (\alpha \supset \gamma)]$
Ax_3: $(\sim\alpha \supset \sim \beta) \supset (\beta \supset \alpha)$

Rule of substitution: Any wff which results from consistently substituting wffs for each of the terms in any of the axioms above is a theorem.

The same logical truths that we can prove in Hurley's system are provable in this system.
For example, consider the tautology '$P \supset P$'.
We have not seen how to prove this in Hurley's system, yet.
But, here is a proof in the axiom system, above.

1. $P \supset [(P \supset P) \supset P]$	Ax_1
2. $\{P \supset [(P \supset P) \supset P]\} \supset \{[P \supset (P \supset P)] \supset (P \supset P)\}$	Ax_2
3. $[P \supset (P \supset P)] \supset (P \supset P)$	1, 2, MP
4. $P \supset (P \supset P)$	Ax_1
5. $P \supset P$	3, 4, MP
QED	

Here is another one, for '$(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$'

1. $[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$	Ax_2
2. $\{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}\}$	Ax_1
3. $(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}$	1, 2, MP
4. $\{(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}\} \supset \{\{(Q \supset R) \supset [P \supset (Q \supset R)]\} \supset \{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\}\}$	Ax_2
5. $\{(Q \supset R) \supset [P \supset (Q \supset R)]\} \supset \{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\}$	4, 3, MP
6. $(Q \supset R) \supset [P \supset (Q \supset R)]$	Ax_1
7. $(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]$	5, 6, MP
8. $\{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}$	Ax_2
9. $[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]$	8, 7, MP
10. $\{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\} \supset \{(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}\}$	Ax_1
11. $(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}$	10, 9, MP
12. $\{(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}\} \supset \{\{(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]\} \supset \{(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]\}\}$	Ax_2
13. $\{(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]\} \supset \{(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]\}$	12, 11, MP
14. $(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]$	Ax_1
15. $(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$	13, 14, MP

QED

If you want to try one, you can do '$(\sim P \supset P) \supset P$' in about 11 lines.
'$[P \supset (Q \supset R)] \supset [Q \supset (P \supset R)]$' is not too long, either.
Note that in such systems, every line of the proof is a law of logic.
In the proofs we have been doing, almost none of the lines are laws of logic, since they mostly are, or rest on, assumptions.
We can add assumptions into axiom systems, as well.
By adding small, particular assumptions, we can develop all of mathematics.

V. On papers

There are three kinds of papers you can write, for this course

1. Logic and philosophy
2. Philosophy of logic
3. Logic

Our discussion of propositions can take you toward a philosophical topic, about abstract objects.
Last Philosophy Friday, when we explored three-valued logics, we looked at some philosophical issues in logic that might impel one toward a three-valued logic.
And, at the end of today, and last Philosophy Friday, we looked at some of the technical work.

You should be thinking already about choosing a topic and putting together a draft of a paper.
Don't forget to look at the course bibliography.