Philosophy 240: Symbolic Logic Hamilton College
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Class 15: Rules of Replacement, I (§7.3)

I. **The First Five Rules of Replacement**

Rules of Inference allow you to derive new conclusions based on previously accepted premises or derivations.
They are justified by appeal to the truth-table definitions of validity.
So, they must be used on whole lines only, and they go only one-way.

Rules of Replacement allow you to substitute one proposition or part of a proposition with a logically equivalent expression.
Since they are based on truth-table rules of equivalence, they may be used for any expressions, any where in a proof.
They may be used on parts of lines, or on whole lines.
They may be used in either direction.
To check the legitimacy of the substitutions, we must use truth tables to show that the expressions are in fact logically equivalent.
See the appendix at the end of the lesson.

DeMorgan's Laws (DM)

$\sim(\alpha \cdot \beta) :: \sim\alpha \vee \sim\beta$
$\sim(\alpha \vee \beta) :: \sim\alpha \cdot \sim\beta$

Note the use of '::' to mean 'is logically equivalent to'.
It is a symbol of the metalanguage, like the Greek letters, and not a symbol of the language of propositional logic.
Also note that there are two versions of DM: one for the negation of a conjunction, and the other for the negation of a disjunction.
Like all Rules of Replacement, you can go either way.
Forward is like distribution in algebra.
Backward is like factoring out the negation.
Note that both the forward and the backward uses require the same justification.

Sample Derivations
A Forward use:

1. $(A \vee B) \supset E$
2. $\sim E$
3. $A \vee D$ / D
4. $\sim(A \vee B)$ 1, 2, MT
5. $\sim A \cdot \sim B$ 4, DM
6. $\sim A$ 5, Simp
7. D 3, 6, DS

QED

A Backward use:

1. $G \supset (H \cdot F)$
2. $\sim H \vee \sim F$ / $\sim G$
3. $\sim(H \cdot F)$ 2, DM
4. $\sim G$ 1, 3, MT

QED

Associativity (Assoc)

$\alpha \vee (\beta \vee \gamma) :: (\alpha \vee \beta) \vee \gamma$

$\alpha \cdot (\beta \cdot \gamma) :: (\alpha \cdot \beta) \cdot \gamma$

Assoc is often used with DS.
Again, there is a conjunction version, and there is a disjunction version.
Note that the two connectives must be the same.

Sample Derivation:

1. $(L \vee M) \vee N$
2. $\sim L$
3. $(M \vee N) \supset O$ / O
4. $L \vee (M \vee N)$ 1, Assoc
5. $M \vee N$ 4, 2, DS
6. O 3, 5, MP

QED

Distributivity (Dist)

$\alpha \cdot (\beta \vee \gamma) :: (\alpha \cdot \beta) \vee (\alpha \cdot \gamma)$

$\alpha \vee (\beta \cdot \gamma) :: (\alpha \vee \beta) \cdot (\alpha \vee \gamma)$

Again, there are two versions: distributing the conjunction over the disjunction and distributing the disjunction over the conjunction.
Note that the order of the connectives remains the same, with an extra of the first connective added at the end (or taken away).
so $\cdot\vee$ becomes $\cdot\vee\cdot$
and $\vee\cdot$ becomes $\vee\cdot\vee$
(or vice versa)
Using it on ‘$P \vee (Q \cdot R)$’ yields a conjunction, from which you can simplify!
Assoc is used when you have two of the same connectives, Dist is used when you have a combination of conjunction and disjunction.

Sample Derivation (forward):

1. $H \cdot (I \vee J)$
2. $\sim(H \cdot I)$ / $H \cdot J$
3. $(H \cdot I) \vee (H \cdot J)$ 1, Dist
4. $H \cdot J$ 3, 2, DS

QED

Sample Derivation (backward):

1. $(P \vee Q) \cdot (P \vee R)$
2. $\sim P$ $/ Q \cdot R$
3. $P \vee (Q \cdot R)$ 1, Dist
4. $Q \cdot R$ 3, 2, DS

QED

Commutativity (Com)

$\alpha \vee \beta :: \beta \vee \alpha$

$\alpha \cdot \beta :: \beta \cdot \alpha$

You may combine a use of Com with other rules on the same line.
In effect, it doubles the rules DS, Simp, and Add.

Now we can derive 'P' from 'P ∨ Q' and '~Q':

1. $P \vee Q$
2. $\sim Q$
3. $Q \vee P$ 1, Com
4. P 3, 4, DS   (Lines 3 and 4 may be combined.)

Also, we can derive 'Q' from 'P · Q'

1. $P \cdot Q$
2. $Q \cdot P$ 1, Com
3. Q 2, Simp   (Lines 2 and 3 may be combined.)

Also, we can derive 'Q ∨ P' from 'P'

1. P
2. $P \vee Q$ 1, Add
3. $Q \vee P$ 2, Com

Each of these three above derivations can be inserted into any derivation by substituting appropriately.

Sample Derivation:

1. $A \cdot B$
2. $B \supset (D \vee E)$
3. $\sim E$ $/ D$
4. B 1, Com, Simp
5. $D \vee E$ 2, 4, MP
6. D 5, 3, Com, DS

QED

Double Negation (DN)

α :: ~~α

There are three ways to double-negate a statement with a binary connective. E.g. consider 'P ∨ Q'. It can be turned into:

1. ~~P ∨ Q — by double negating the 'P'
2. P ∨ ~~Q — by double negating the 'Q'
3. ~~(P ∨ Q) — by double negating the '∨'

Sample Derivation:

1. ~F ⊃ ~G	
2. G	
3. F ⊃ H	/ H
4. ~~F	1, 2, DN, MT
5. H	3, 4, DN, MP
QED	

II. **Using Replacement Rules on Part of Lines**

Rules of replacement apply to any part of a proof, not just whole lines.
This does not apply to Rules of Inference.
The following inference is not allowed, directly, though it can be derived through several steps:

P ⊃ (Q ⊃ R)
Q
P ⊃ R

We can using DM on part of a line.

P ⊃ ~(Q ∨ P) can be transformed to
P ⊃ (~Q · ~P)

We can use DN on part of a line, too.

~P · Q	can be transformed to:
~P · ~~Q	which can then be transformed, e.g., to:
~(P ∨ ~Q)	using DM

A similar equivalence can be shown by switching conjunction and disjunction in the above.

III. **Exercises**. Derive the conclusions of each of the following arguments using the rules of inference and the first five rules of replacement.

1.
 1. A ∨ B
 2. ~D ∨ E
 3. ~(A ∨ E) / B · ~D

2.
 1. P ∨ (Q · R)
 2. ~Q / P

3.
 1. ~(S ∨ T)
 2. U ⊃ T
 3. W ⊃ U / ~W

4.
 1. ~(F · G)
 2. ~(F · H)
 3. G ∨ H / ~F

5.
 1. A ∨ (B · ~F)
 2. (A ⊃ ~D) · (~F ⊃ ~E) / ~(D · E)

6.
 1. (H ∨ I) ⊃ (J · K)
 2. ~J / ~I

Solutions may vary.

IV. **Appendix: Proofs of the Logical Equivalence of the First Three Rules of Replacement**

DeMorgan's Rules: ~(α ∨ β) :: ~α · ~β

~	(α	∨	β)
F	T	T	T
F	T	T	F
F	F	T	T
T	F	F	F

~	α	·	~	β
F	T	**F**	F	T
F	T	**F**	T	F
T	F	**F**	F	T
T	F	**T**	T	F

DeMorgan's Rules: ~(α · β) :: ~α ∨ ~β

~	(α	·	β)
F	T	T	T
T	T	F	F
T	F	F	T
T	F	F	F

~	α	∨	~	β
F	T	**F**	F	T
F	T	**T**	T	F
T	F	**T**	F	T
T	F	**T**	T	F

Associativity: α ∨ (β ∨ γ) :: (α ∨ β) ∨ γ

α	∨	(β	∨	γ)
T	**T**	T	T	T
T	**T**	T	T	F
T	**T**	F	T	T
T	**T**	F	F	F
F	**T**	T	T	T
F	**T**	T	T	F
F	**T**	F	T	T
F	**F**	F	F	F

(α	∨	β)	∨	γ
T	T	T	**T**	T
T	T	T	**T**	F
T	T	F	**T**	T
T	T	F	**T**	F
F	T	T	**T**	T
F	T	T	**T**	F
F	F	F	**T**	T
F	F	F	**F**	F

Associativity: α · (β · γ) :: (α · β) · γ

α	·	(β	·	γ)
T	**T**	T	T	T
T	**F**	T	F	F
T	**F**	F	F	T
T	**F**	F	F	F
F	**F**	T	T	T
F	**F**	T	F	F
F	**F**	F	F	T
F	**F**	F	F	F

(α	·	β)	·	γ
T	T	T	**T**	T
T	T	T	**F**	F
T	F	F	**F**	T
T	F	F	**F**	F
F	F	T	**F**	T
F	F	T	**F**	F
F	F	F	**F**	T
F	F	F	**F**	F

Distributivity: α ∨ (β · γ) :: (α ∨ β) · (α ∨ γ)

α	∨	(β	·	γ)
T	**T**	T	T	T
T	**T**	T	F	F
T	**T**	F	F	T
T	**T**	F	F	F
F	**T**	T	T	T
F	**F**	T	F	F
F	**F**	F	F	T
F	**F**	F	F	F

(α	∨	β)	·	(α	∨	γ)
T	T	T	**T**	T	T	T
T	T	T	**T**	T	T	F
T	T	F	**T**	T	T	T
T	T	F	**T**	T	T	F
F	T	T	**T**	F	T	T
F	T	T	**F**	F	F	F
F	F	F	**F**	F	T	T
F	F	F	**F**	F	F	F

Distributivity: α · (β ∨ γ) :: (α · β) ∨ (α · γ)

α	·	(β	∨	γ)
T	**T**	T	T	T
T	**T**	T	T	F
T	**T**	F	T	T
T	**F**	F	F	F
F	**F**	T	T	T
F	**F**	T	T	F
F	**F**	F	T	T
F	**F**	F	F	F

(α	·	β)	∨	(α	·	γ)
T	T	T	**T**	T	T	T
T	T	T	**T**	T	F	F
T	F	F	**T**	T	T	T
T	F	F	**F**	T	F	F
F	F	T	**F**	F	F	T
F	F	T	**F**	F	F	F
F	F	F	**F**	F	F	T
F	F	F	**F**	F	F	F