Philosophy 240: Symbolic Logic Hamilton College
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Class 12: Rules of Implication (§7.1)

I. **A System of Natural Deduction**

Natural deduction is a method of proof of validity.
We use the Language of Propositional Logic, plus 8 Rules of Inference, and 10 Rules of Replacement.
Once we've proven a rule valid, using the indirect truth table method, then we can use it in a proof.
A proof is a sequence of wffs every member or which either is an assumption or follows from earlier wffs in the sequence according to specified rules of inference.
We can prove that each of the rules of inference preserves truth.
A rule of inference preserves truth if every substitution instance, every formula of that form, is valid.
The rules of inference are purely syntactic, which means that they preserve truth, no matter how we interpret the propositional variables.

Deductions generally begin with any number of premises, and end with a conclusion.
A deduction is valid if every step is either a premise or derived from premises or previous steps using valid rules of inference.

II. **Some Rules of Inference**

Consider the validity of each of the following:

1. $A \supset B$
   $A \qquad / B$

2. $(E \cdot I) \supset D$
   $(E \cdot I) \qquad / D$

3. $\sim G \supset (F \cdot H)$
   $\sim G \qquad / F \cdot H$

Note that they share their common (valid) form:

$\alpha \supset \beta$
$\alpha \qquad / \beta$

This form is called *Modus Ponens*, and abbreviated (MP).
For example, "If I own a Toyota, then I own a car. I own a Toyota. So, I own a car."
Note that we can substitute simple or complex formulae for P and Q.
Another example of MP:

$[(H \vee G) \supset I] \supset (K \cdot \sim L)$
$[(H \vee G) \supset I] \qquad / (K \cdot \sim L)$

The following three forms are also valid.
We can check them, using indirect truth tables.

$\alpha \supset \beta$
$\sim\beta \qquad / \sim\alpha$

This form is called *Modus Tollens*, abbreviated MT.
For example, "If I own a Toyota, I own a car. I don't own a car. So, I don't own a Toyota."

$\alpha \vee \beta$
$\sim\alpha \qquad / \beta$

This form is called *Disjunctive Syllogism*, abbreviated DS.
For example, "I will have soup or salad. I don't have soup. So, I will have salad."

$\alpha \supset \beta$
$\beta \supset \gamma \qquad / \alpha \supset \gamma$

This form is called *Hypothetical Syllogism*, abbreviated HS.
For example, "If I own a Toyota I own a car. If I own a car, I have to pay for insurance. So, If I own a Toyota, I have to pay for insurance."
The following two forms are invalid.
Again, we can check them, using truth tables, or indirect truth tables.

$\alpha \supset \beta$
$\beta \qquad / \alpha$

This form is called the *Fallacy of Affirming the Consequent*.
For example, "If I own a Toyota, I own a car. I own a car. So I own a Toyota."
Note that the premises may be true while the conclusion is false.

$\alpha \supset \beta$
$\sim\alpha \qquad / \sim\beta$

This form is called the *Fallacy of Denying the Antecedent*.
For example, "If I own a Toyota, I own a car. I don't own a Toyota. So, I don't own a car.

III. **Examples of deductions**

Show that the following argument is valid:

1. $(X \supset Y) \supset T$
2. $S \vee \sim T$
3. $U \supset \sim S$
4. $U \qquad / \sim(X \supset Y)$

We can use truth tables, but we can now also use the method of natural deduction.

1. $(X \supset Y) \supset T$		
2. $S \vee {\sim}T$		
3. $U \supset {\sim}S$		
4. U	/ ${\sim}(X \supset Y)$	
5. ${\sim}S$	3, 4, MP	(taking ‘U’ for P and ‘~S’ for Q)
6. ${\sim}T$	2, 5, DS	(taking ‘S’ for P and ‘~T’ for Q)
7. ${\sim}(X \supset Y)$	1, 6, MT	(taking ‘X ⊃ Y’ for P and ‘T’ for Q)
QED		

Notes on the above deduction:

All lines except the premises require justification, which includes the lines and rule of inference used to generate the new conclusion. For example, ‘3, 4, MP’ means that the current line is derived directly from lines 3 and 4 by a use of the rule of Modus Ponens.

The conclusion, written after a single slash following the last premise is not technically part of the deduction.

Deductions are sometimes called proofs, and sometimes called derivations.

The explanations such as “taking ‘U’ for P and ‘~S’ for Q” are not required elements of the derivation.

‘QED’ stands for ‘Quod erat demonstratum’, meaning ‘Thus it has been shown’, and serves as a logician’s punctuation mark: “I’m done!” It is not required, but looks neat.

Another example:

1. ${\sim}G \supset [G \vee (S \supset A)]$	
2. $(S \vee L) \supset {\sim}G$	
3. $S \vee L$	
4. $A \supset G$	/ L
5. ${\sim}G$	2, 3, MP
6. $G \vee (S \supset A)$	1, 5, MP
7. $S \supset A$	6, 5, DS
8. $S \supset G$	7, 4, HS
9. ${\sim}S$	8, 5, MT
10. L	3, 9, DS
QED	

Some hints for constructing a derivation:

Start with simple sentences, or negations of simple negations.
Plan ahead, work backwards on the side.
Don’t worry about extraneous lines: not every line must be used.
Some lines may be used more than once.

IV. **Exercises**. Derive the conclusions of each of the following arguments using natural deduction.

1.
1. $(A \cdot B) \supset (E \vee D)$
2. $A \cdot B$
3. $\sim E$ / D

2.
1. $\sim D \vee (H \vee F)$
2. $H \supset G$
3. $\sim\sim D$
4. $\sim G$ / F

3.
1. $X \supset Y$
2. $\sim Z$
3. $Y \supset Z$
4. $X \vee W$ / W

4.
1. $A \supset \sim B$
2. $A \vee (D \supset E)$
3. $\sim B \supset E$
4. $\sim E$ / $\sim D$

Solutions may vary.