Philosophy 240: Symbolic Logic Hamilton College
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Class 10: Three-Valued Logics
Fisher, pp 36-39; 125-131

I. Seven motivations for three-valued logics

M1. Mathematical sentences with unknown truth values
Consider:

A. Every even number greater than four can be written as the sum of two odd primes.

A is called Goldbach's conjecture, though Euler seems to have formulated it in response to a weaker hypothesis raised by Goldbach in 1742.
Goldbach's conjecture has neither been proven true nor false, though it has been verified up to very large values.
There are, also, inductive arguments which make mathematicians pretty confident that Goldbach's conjecture is true.
But, some smart mathematicians have tried and failed to come up with a proof.
We might take Goldbach's conjecture to be neither true nor false.
We might do so, especially, if we think that mathematics is constructed, rather than discovered.
If and when some one proves it, or its negation, then we could apply a truth value to the proposition.
Until we have a proof, we might take Goldbach's conjecture to lack a truth value.

M2. Statements about the future
Consider:

B. There will be a party tomorrow night on the Dunham Quad.

The problem in B may be found in Aristotle's *De Interpretatione*, §9, regarding a sea battle.
The problem is generally labeled that of future contingents.
Since the statement is contingent, we can neither assert its truth nor its falsity.
We know that one of those values will apply, eventually.
But, right now, it seems to lack a truth value.
If it had a truth value now, then the event would not be contingent; it would already be determined.
Since the future is not determined, the truth values of statements about the future should also be undetermined.

M3. Failure of presupposition
Consider:

C. The king of America is bald.

C is an example of failure of presupposition.
Contrast C with

D: The king of America is not bald.

Neither C nor D are true propositions.
But, D looks like the negation of C.
That is, if we regiment C as ‘P’, we should regiment D as ‘~P’.
(‘Regiment’ means ‘translate into logic’.)
Now, since ‘P’ is not true, we must call it false, since we have only two truth values.
Assigning the value ‘false’ to ‘P’ means that ‘~P’ should take the ‘true’.
Uh-oh.
The problem is that we want both a proposition and its negation to be false, but the negation of a false proposition is a true proposition.
Why do we think that C and D are both false?
They both contain a false presupposition.
The following sentences all contain a failure of presupposition.

E: The woman on the moon is six feet tall.
F: The rational square root of three is less than two.
G: When did you stop beating your wife?

Sentence G is not declarative, but also contains a failure of presupposition.
One response to the problem of presupposition failure in propositions is to call such propositions neither true nor false.

M4. Semantic paradoxes
Consider:

H. ‘H’ is false.

H is an example of a paradoxical sentence.
If H is true, then it is false, which makes it true, which makes it false...
H seems to lack a definite truth value, even though it is a perfectly well-formed sentence.
It is often called Epimenides paradox.
Epimenides was a Cretan to whom the statement that all Cretans are liars is attributed.
There are grounds to question the paradoxicality of H, which do not extend to I.

I. ‘Yields falsehood when appended to its own quotation’ yields falsehood when appended to its own quotation.

I comes from Quine’s essay, ‘The Ways of Paradox’.
For another example of a paradoxical sentence, consider Grelling’s paradox.
Some predicates apply to themselves, whereas others do not.
‘Polysyllabic’ is polysyllabic; ‘monosyllabic’ is not monosyllabic.
Call a predicate heterological if it does not apply to itself.
‘Monosyllabic’ is heterological; ‘polysyllabic’ is not heterological.
(We can call it autological, or homological.)
Now, consider whether ‘heterological’ applies to itself.
If it does, then ‘heterological’ is not heterological.
But, if ‘heterological’ is not heterological, then it does not apply to itself, which means that it is heterological.
Here we go again!

We can construct a statement involving ‘heterological’ whose truth value is puzzling.

J. ‘Heterological’ is heterological.

Here is one more paradox, because they are fun.
This one is due to Bertrand Russell, though he credits an anonymous source.
Consider the barber in a town who shaves all the men who do not shave themselves.
Does he shave himself?
Again, you can construct a puzzling declarative sentence, which I leave to you as an exercise.
One response to such paradoxical sentences is to say that they lack a truth value, or to try to assign a third truth value.
Here are three other motivations, briefly:

M5. Nonsense
We might consider some grammatical but nonsensical sentences to be truth-valueless.

K. Quadruplicity drinks procrastination. (From Bertrand Russell)
L. Colorless green ideas sleep furiously. (From Noam Chomsky)

M6. Vagueness
Many predicates admit of borderline, or vague, cases.

M. My minivan is a car.
N. It is a nice day.

Some days are clearly nice days, others are clearly not nice, but in the middle, there is a penumbra.

M7. Programming needs
In computer programming, we might want to leave the values of some variables unassigned, during a process.
So, we might want a logic which uses a third value.
M7 is merely a pragmatic motivation, rather than a philosophical one.
It is, though, by itself reason to explore the technical merits of three-valued logics.
Then, we can wonder about the philosophical motivations and applications.

M8. The paradoxes of the material conditional
See Class 5, on conditionals.

III. Three-valued logics

The rules for determining truth values of formulas in a logic are called the semantics.
We provided semantics for propositional logic by constructing truth tables.
Since we used only two values, true and false, our semantics is called two-valued.
Our two-valued semantics is also called classical semantics
If we want to adopt a third truth value, which we might call unknown, or indeterminate, we must revise all the truth tables.
Because I can not make up my mind about which name to use, I will call the third truth value indeterminate, and use the letter 'U' to indicate it.

Remember, the idea is that we can ascribe 'U' to sentences like:

A. Every even number greater than four can be written as the sum of two odd primes.
B. There will be a party tomorrow night on the Dunham Quad.
C. The king of America is bald.
D: The king of America is not bald.
E: The woman on the moon is six feet tall.
F: The rational square root of three is less than two.
G: When did you stop beating your wife?
H. 'H' is false.
I. 'Yields falsehood when appended to its own quotation' yields falsehood when appended to its own quotation.
J. 'Heterological' is heterological.
K. Quadruplicity drinks procrastination. (From Bertrand Russell)
L. Colorless green ideas sleep furiously. (From Noam Chomsky)
M. My minivan is a car.
N. It is a nice day.

There are two options for how to deal with unknown or indeterminate truth values in the new semantics.
First, one could claim that any indeterminacy among component propositions creates indeterminacy in the whole.
This the principle underlying Bochvar's semantics.
Second, one could try to ascribe truth values to as many formulas as possible, despite the indeterminate truth values.

We proceed to look at three different three-valued semantics.
We will look at:

1. The rules for each;
2. How the new rules affect the logical truths (tautologies); and
3. How the new rules affect the allowable inferences (valid arguments).

Bochvar semantics

P	~P
T	F
U	U
F	T

P	•	Q
T	T	T
T	U	U
T	F	F
U	U	T
U	U	U
U	U	F
F	F	T
F	U	U
F	F	F

P	∨	Q
T	T	T
T	U	U
T	T	F
U	U	T
U	U	U
U	U	F
F	T	T
F	U	U
F	F	F

P	⊃	Q
T	T	T
T	U	U
T	F	F
U	U	T
U	U	U
U	U	F
F	T	T
F	U	U
F	T	F

For simplicity, we ignore the biconditional, which is definable in terms of the other connectives anyway.
In Bochvar semantics, no classical tautologies come out as tautologies.
Consider ‘P ⊃ P’, under Bochvar semantics:

P	⊃	P
T	T	T
U	U	U
F	T	F

Or, ‘P ⊃ (Q ⊃ P)’ under Bochvar:

P	⊃	(Q	⊃	P)
T	T	T	T	T
T	U	U	U	T
T	T	F	T	T
U	U	T	U	U
U	U	U	U	U
U	U	F	U	U
F	T	T	F	F
F	U	U	U	F
F	T	F	T	F

These classical tautologies, and all others, do not come out false on any line on Bochvar semantics.
But, they do not come out as true on every line.
This result is generally undesirable, since the classical tautologies seem pretty solid.
Tautologies are also known as logical truths.
They are the theorems of the logic.

Some people think that some of the classical tautologies are suspect anyway.
In particular, ‘P ⊃ (Q ⊃ P)’ seems counter-intuitive.
It is sometimes called a paradox of material implication.
Other systems of logic, called relevance logics, attempt to keep most classical logical truths, but eliminate the paradoxes of material implication.
Unfortunately, Bochvar semantics seems to throw the baby out with the bath water, in eliminating all classical tautologies.

One solution to the problem of losing logical truths in Bochvar semantics would be to redefine ‘tautology’ as a statement which never comes out as false.
Redefining ‘tautology’ in this way, though, weakens the concept, making it less useful.
Quine, in *Philosophy of Logic*, Chapter 6, calls three-valued logics “deviant”, and urges that they constitute a change of topic, rather than an improvement of logic.

Now, consider what Bochvar semantics does to validity.
We defined a valid argument as one for which there is no row in which the premises are true and the conclusion is false.
We could have defined a valid argument as one for which there is no row in which the premises are true and the conclusion is not true.
Classically, these two definitions are equivalent.
But, in three-valued semantics, they cleave.
If we take a row in which the premises are true and the conclusion is indeterminate as a counterexample to an argument, as Bochvar did, then some classically valid inferences come out invalid.

Consider: 'P /∴ Q ∨ P'
Under classical semantics, this argument is valid.

P	/	Q	∨	P
T		T	**T**	T
T		F	**T**	T
F		T	**T**	F
F		F	**F**	F

Now, look at the same argument under Bochvar semantics:

P	/	Q	∨	P
T		T	**T**	T
T		U	**U**	T
T		F	**T**	T
U		T	**U**	U
U		U	**U**	U
U		F	**U**	U
F		T	**T**	F
F		U	**U**	F
F		F	**F**	F

The second row is now a counterexample!

Bochvar semantics proceeds on the presupposition that any indeterminacy infects the whole.
It thus leaves the truth values of many formulas undetermined.
But, we might be able to fill in some of the holes.

That is, why should we consider the disjunction of a true statement with one of indeterminate truth value to be undetermined?
Or, why should we consider the conditional with an antecedent of indeterminate truth value to itself be of indeterminate truth value, if the consequent is true?
Whatever other value we can assign the variables with unknown truth value, both sentences will turn out to be true.
Kleene's semantics leaves fewer rows unknown.
(The semantics that follows is sometimes called strong Kleene, to distinguish from Bochvar, which is sometimes called weak Kleene.)

Kleene semantics

P	~P
T	F
U	U
F	T

P	•	Q
T	T	T
T	U	U
T	F	F
U	U	T
U	U	U
U	**F**	F
F	F	T
F	**F**	U
F	F	F

P	∨	Q
T	T	T
T	**T**	U
T	T	F
U	**T**	T
U	U	U
U	U	F
F	T	T
F	U	U
F	F	F

P	⊃	Q
T	T	T
T	U	U
T	F	F
U	**T**	T
U	U	U
U	U	F
F	T	T
F	**T**	U
F	T	F

Kleene semantics has a certain intuitiveness.
But, in order to compare Bochvar to Kleene properly, we should look at the differences on logical truths and inference patterns.

Consider the same two tautologies, ‘P ⊃ P’ and ‘P ⊃ (Q ⊃ P)’ under Kleene semantics:

P	⊃	P
T	**T**	T
U	**U**	U
F	**T**	F

P	⊃	(Q	⊃	P)
T	**T**	T	T	T
T	**T**	U	T	T
T	**T**	F	T	T
U	**U**	T	U	U
U	**U**	U	U	U
U	**T**	F	T	U
F	**T**	T	F	F
F	**T**	U	U	F
F	**T**	F	T	F

While many more of the rows are completed, the statements still do not come out as tautologous, under the classical definition of ‘tautology’.
Lukasiewicz, who first investigated three-valued logics, tried to preserve the tautologies.
There is only one difference between Kleene semantics and Lukasiewicz semantics.

Lukasiewicz semantics

P	~P
T	F
U	U
F	T

P	•	Q
T	T	T
T	U	U
T	F	F
U	U	T
U	U	U
U	F	F
F	F	T
F	F	U
F	F	F

P	∨	Q
T	T	T
T	T	U
T	T	F
U	T	T
U	U	U
U	U	F
F	T	T
F	U	U
F	F	F

P	⊃	Q
T	T	T
T	U	U
T	F	F
U	T	T
U	**T**	U
U	U	F
F	T	T
F	T	U
F	T	F

One might wonder how we might justify calling a conditional with indeterminate truth values in both the antecedent and consequent true.
For, what if the antecedent turns out true and the consequent turns out false?

Put that worry aside, and look at what this one small change does:

P	⊃	P
T	**T**	T
U	**T**	U
F	**T**	F

P	⊃	(Q	⊃	P)
T	**T**	T	T	T
T	**T**	U	T	T
T	**T**	F	T	T
U	**T**	T	U	U
U	**T**	U	T	U
U	**T**	F	T	U
F	**T**	T	F	F
F	**T**	U	U	F
F	**T**	F	T	F

Voila!
We retain many of the classical tautologies!
In fact, we do not get all classical tautologies.
Consider the law of excluded middle, 'P ∨ ~P'

P	∨	~	P
T	**T**	F	T
U	**U**	U	U
F	**T**	T	F

Excluded middle still does not come out tautologous.
But, that is a law that some folks would like to abandon, anyway.
But, is the change motivated?
The lesson of Lukasiewicz semantics is that we need not give up classical tautologies, logical truths, to have a three-valued logic.
The fewer changes we make to the set of logical truths, the less "deviant" the logic is.
But, the semantics which allows us to retain these logical truths may not be as pretty as we would like.

Lastly, consider the effect on validity of moving from Bochvar to Kleene or Lukasiewicz.
Consider again the argument: ‘P /∴ Q ∨ P:

Bochvar:

P	/	Q	∨	P
T		T	**T**	T
T		U	**U**	T
T		F	**T**	T
U		T	**U**	U
U		U	**U**	U
U		F	**U**	U
F		T	**T**	F
F		U	**U**	F
F		F	**F**	F

counter-example in Row 2

Kleene:

P	/	Q	∨	P
T		T	**T**	T
T		U	**T**	T
T		F	**T**	T
U		T	**T**	U
U		U	**U**	U
U		F	**U**	U
F		T	**T**	F
F		U	**U**	F
F		F	**F**	F

valid - no counter-example

Lukasiewicz:

P	/	Q	∨	P
T		T	**T**	T
T		U	**T**	T
T		F	**T**	T
U		T	**T**	U
U		U	**U**	U
U		F	**U**	U
F		T	**T**	F
F		U	**U**	F
F		F	**F**	F

valid - no counter-example

Both Kleene and Lukasiewicz semantics thus maintain some of the classical inference patterns.
See the exercises in §VI of these notes for more comparisons.

IV. Problems with three-valued logics

We have already discussed the loss of (at least some) classical tautologies and classically valid inference patterns.
Furthermore, it is not clear that all the problems that motivated three-valued logics can be solved by three-valued logics.
For example, Bochvar hoped that his semantics would solve the problems of the semantic paradoxes.
Sentence H, above, can be given a truth value in Bochvar semantics without paradox.
But, consider

O: ‘O’ is untrue

Now, suppose ‘O’ is true.
Thus, ‘O’ is untrue.
But then ‘O’ turns out to be true (because it says that ‘O’ is untrue).
And here we go again!

Another worry about three-valued logics is that assigning a truth-value of ‘unknown’ involves a conceptual confusion.
‘Unkown’ may not be a third truth value, but merely the lack of a truth value.
Instead of filling in such cells in the truth table, we should just leave them blank.
Leaving certain cells of the truth table blank is part of what is called the truth-value gap approach.
Faced with truth-value gaps, or partial valuations, the logician may consider something called a supervaluation.
A supervaluation considers the different ways to complete partial valuations, and classifies formulas and arguments according to the possibilities for completion.
But, that is a topic for another time.

For more worries about three-valued logics, see paper topic 4, below.
In particular, Quine’s worry about the deviance of three-valued logic is both profound and enormously influential.

V. Avoiding three-valued logics

We started introducing three-valued logics in order to respond to some problems which arose with classical logic.

M1. Mathematical sentences with unknown truth values
M2. Statements about the future
M3. Failure of presupposition
M4. Semantic paradoxes
M5. Nonsense
M6. Vagueness
M7. Programming needs

I mentioned that three-valued logics does not solve the problems of the semantic paradoxes, M4.
There are ways for the classical logician to deal with all of these problems, anyway.
I will not discuss each of them, here.
But, here are a few hints to how to solve them.

M1, concerning sentences with unknown truth values and M2, concerning propositions referring to future events, are related.
In both cases, we can blame ourselves, rather than the world, for our not knowing the truth value.
Thus, we can say that Goldbach’s conjecture is either true or false, but we just do not know which.
Similarly, we can say that either there will be a party at Dunham tomorrow, or there will not.
We need not ascribe a deep problem to truth values.
Such sentences have truth values.
We just do not know them.
Classical logic deals with problems about time by appealing to a four-dimensional framework.
We take a God’s-eye point of view.
Going four-dimensional, we add a time-stamp to all our claims.
Then, a statement about the future is true if it ends up true at the time.
We need not see the logic as committing us to a determined future.
We just know that statements about future events will eventually have truth values.

There are also tense logics, which introduce temporal operators but maintain classical semantics, to help with time.

For failures of presupposition, M3, we can use Bertrand Russell's analysis of definite descriptions.
In brief, we analyze the sentence to make the assumption explicit.
In the last unit of this course, we will see a more precise analysis of Russell's solution.
For now, roughly, recast sentence E, 'The woman on the moon is six feet tall', as:

Y: There is a woman on the moon and she is six feet tall.

Sentence Y has the form 'P • Q'.
'P' is false, so 'P • Q' is false.
We can similarly regiment 'The woman on the moon is not six feet tall'.

Z: 'There is a woman on the moon and she is not six feet tall'

We regiment sentence Z as 'P • ~Q'
P is false, so 'P • ~Q' is false.
We thus do not have a situation in which the same proposition seems true and false.
In both cases, P is false, so the account of the falsity of both sentences Y an Z can be the same.
We thus lose the motivation for introducing a third truth value.

Lastly, for M4, paradoxes, M5, nonsense, and M6, vagueness, we can deny that such sentences express propositions.
We may claim that just as some strings of letters do not form words, and some strings of words do not form sentences, some grammatical sentences do not express propositions.
This would be the same as to call them meaningless.
This solution is a bit awkward, since it does seem that 'This sentence is false' is perfectly meaningful.
But if it prevents us from having to adopt three-valued logics, it might be a useful move.

VI. Some exercises you might try:

Construct truth tables for each of the following propositions, under classical semantics and each of the three three-valued semantics (Bochvar, Kleene, Lukasiewicz). Compare the results. Note: these exercises might form some part of a paper on three-valued semantics.

1. P ∨ ~P
2. P ⊃ P
3. (P ⊃ Q) ≡ (~P ∨ Q)

   Note: you can construct the truth table for the biconditional by remembering that 'P≡Q' is logically equivalent to '(P ⊃ Q) • (Q ⊃ P)'

Use the indirect method of truth tables to test each of the following arguments for validity, under classical semantics and each of the three three-valued semantics (Bochvar, Kleene, Lukasiewicz). Compare the results.

1. $P \supset Q$
   $P \quad /\therefore Q$

2. $P \quad /\therefore \sim(Q \bullet \sim Q)$

3. $P \quad /\therefore P \vee Q$

VII. A few paper ideas:

1. Compare Bochvar semantics, Kleene semantics, and Lucasiewicz semantics. What differences do the different semantics have for classical tautologies? What differences do they have for classical inferences (validity and invalidity)? Be sure to consider the semantics of the conditional. Which system seems most elegant? This paper will be mainly technical, explaining the different semantics and their results.

2. Do three-valued logics solve their motivating problems? Philosophers explore three-valued logics as a way of dealing with various problems, which I discuss in these notes. Consider some of the problems and show how one of the systems tries to resolve the problem. For this paper, I recommend, but do not insist, that you focus on Kleene's semantics. If you try to deal with Epimenides, and the semantic paradoxes, you might want to focus just on that problem.

3. Bochvar introduced a new operator, which we can label ⊤. Use of this operator allows us to recapture analogs of classical tautologies within Bochvar semantics. Describe the truth table for this operator. Show how it allows us to construct tautologies. How does the new operator affect the set of valid formulas? (It can be shown that on Bochvar semantics, any argument using only the standard operators which has consistent premises, and which contains a sentence letter in the conclusion that does not appear in any of the premises, is invalid. You might consider this result, and the effect of the new operator on it.)

4. Quine, in Chapter 6 of *Philosophy of Logic*, calls three-valued logic deviant, and insists that to adopt three-valued logic is to change the subject. Why does Quine prefer classical logic? Consider his maxim of minimum mutilation. Who can deal better with the problems, sketched at the beginning of these notes, that motivate three-valued logic. (You need not consider all of the problems, but you should provide a general sense of how each approach works.)

There are many more paper ideas leading from topics discussed in these notes. For example, any of the motivations M1-M7 would be a sufficient theme itself. Feel free to meet with me if you want help finding directions.

VIII. Places for further research, in no particular order, except that Quine comes first, as he should

Willard van Orman Quine, *Philosophy of Logic*, 2nd ed. Harvard University Press, 1986. The discussion of deviant logics and changing the subject is in Chapter 6, but Chapter 4, on logical truth, is exceptionally clear and fecund.

Willard van Orman Quine, "The Ways of Paradox," is the title essay in a collection of papers. It is also the source of the 'yields a falsehood...' paradox, and contains an excellent discussion of paradoxes. Harvard University Press, 1976.

Aristotle, *De Interpretatione.* In *The Complete Works of Aristotle, vol.1*, Jonathan Barnes, ed. Princeton University Press, 1984. On the sea battle, and future contingents.

Hilary Putnam, "Three-valued logic" and "The logic of quantum mechanics", in *Mathematics, Matter and Method: Philosophical Papers, vol. 1*. Cambridge University Press, 1975. Do we need three-valued logic in order to account for oddities in quantum mechanics?

Michael Dummett, "The philosophical basis of intuitionist logic". In *Philosophy of Mathematics: Selected Readings*, 2nd ed., Paul Benacerraf and Hilary Putnam, eds. Cambridge University Press, 1983. And the selections by Heyting and Brouwer in the same volume. The intuitionists believed that an unproven mathematical statement lacked a truth value. These articles are all pretty technical, though.

Bertrand Russell, "On Denoting" and "Descriptions". In *The Philosophy of Language*, 5th ed., A.P. Martinich, ed. Oxford University Press, 2008. "On Denoting" is widely available; Martinich includes the slightly less ubiquitous "Descriptions", which has a clearer discussion of Russell's solution to the problem of some forms of failure of presuppopsition.

Timothy Williamson, *Vagueness*. Routledge, 1994. Chapter 1 has a nice discussion of the history of vagueness, and Chapter 4 discusses the three-valued logical approach to the problem.

Noam Chomsky's *Syntactic Structures* contains the discussion about colorless green ideas, but not a defense of three-valued logics. (Chomsky was arguing for a distinction between grammaticality and meaningfulness.)

A.N. Prior, "Three-Valued Logic and Future Contingents" is written in Polish notation.