Hamilton College
Russell Marcus
rmarcus1@hamilton.edu

Class 27: Adequate Sets of Connectives

I. **Theorem 1: The biconditional is superfluous.**
Proof 1A: We just need to show that 'P ≡ Q' and '(P ⊃ Q) • (Q ⊃ P)' are logically equivalent.
We can do this by method of truth tables.

P	≡	Q
T	**T**	T
T	**F**	F
F	**F**	T
F	**T**	F

(P	⊃	Q)	·	(Q	⊃	P)
T	T	T	**T**	T	T	T
T	F	F	**F**	F	T	T
F	T	T	**F**	T	F	F
F	T	F	**T**	F	T	F

QED

II. **Theorem 2: The conditional is superfluous.**
Proof 2A: By method of truth tables.

P	⊃	Q
T	**T**	T
T	**F**	F
F	**T**	T
F	**T**	F

~	P	∨	Q
F	T	**T**	T
F	T	**F**	F
T	F	**T**	T
T	F	**T**	F

Proof 2B: By method of conditional proof.
To show that two statements are logically equivalent, we show that each entails the other.
Assume: 'P ⊃ Q'.
Derive: '~P ∨ Q'.
Then, assume '~P ∨ Q'.
Derive: 'P ⊃ Q'.
QED

Proof 1B: Assume 'P ≡ Q'.
Derive '(P ⊃ Q) • (Q ⊃ P)'.
Assume '(P ⊃ Q) • (Q ⊃ P)'.
Derive 'P ≡ Q'.

QED

III. Two distinct notions of logical equivalence.

LE_1: Two statements are logically equivalent iff they have the same values in every row of the truth table.

LE_2: Two statements are logically equivalent iff each is derivable, using our system of deduction, from the other.

IV. Eliminating both the biconditional and the conditional

Consider: 'Dogs bite if and only if they are startled'.

B ≡ S

(B ⊃ S) • (S ⊃ B)

(~B ∨ S) • (~S ∨ B)

V. Two questions

Q1. How can we be sure that all sentences can be written with just the five connectives?

Q2. Can we get rid of more connectives? What is the fewest number of connectives that we need?

VI. Adequacy defined

A set of connectives is called **adequate** iff corresponding to every possible truth table there is at least one sentence using only those connectives.

VII. **Theorem 3**: Negation and conjunction are adequate, if we use only one propositional variable.

Proof 3: By sheer force.

There are only four possible truth tables: TT, TF, FT, FF

Here are statements for each of them.

~	(P	•	~	P)
T	T	F	F	T
T	F	F	T	F

P
T
F

~	P
F	T
T	F

P	•	~	P
T	**F**	F	T
F	**F**	T	F

QED

VII. Disjunctive Normal Form (DNF) defined

A sentence is in DNF iff if is a series of disjunctions, each disjunct of which is a conjunction of simple letters or negations of simple letters.

A single letter or its negation can be considered a degenerate conjunction or disjunction.

VIII. Exercises A

Pick out which of the following sentences are in DNF:

1. (P • ~Q) ∨ (P • Q)
2. (P • Q • R) ∨ (~P • ~Q • ~R)
3. ~P ∨ Q ∨ R
4. (P ∨ Q) • (P ∨ ~R)
5. (P • Q) ∨ (P • ~Q) ∨ (~P • Q) ∨ (~P • ~R)
6. (~P • Q) • (P • R) ∨ (Q • ~R)
7. (P • ~Q • R) ∨ (Q • ~R) ∨ ~Q
8. ~(P • Q) ∨ (P • R)
9. P • Q
10. ~P

IX. **Theorem 4: The set of negation, conjunction, and disjunction {~, •, ∨} is adequate.**

Proof 4: By cases.

For any size truth table, with any number of connectives, there are three possibilities for the column under the major operator.

Case 1: Every row is false.

Case 2: There is one row which is true, and every other row is false.

Case 3: There is more than one row which is true.

Case 1:

Construct a sentence with one variable in the sentence conjoined with its negation and each of the remaining variables.

So, if you have variables P, Q, R, S, and T, you would write:

(P • ~P) • (Q • S • T)

If you have more variables, add more conjuncts.

The resulting formula, in DNF, is false in every row, and uses only conjunction and negation.

Case 2:

Consider the row in which the statement is true.

Write a conjunction of the following statements:

For each variable, if it is true in that row, write that variable.

For each variable, if it is false in that row, write the negation of that variable.

The resulting formula is in DNF (the degenerate disjunction) and is true in only the prescribed row.

Example:

Consider a formula with two variables: P=TTFF; Q=TFTF;

Major operator=FFFT

We consider the last row only, in which P and Q are both False.

So, we get: '~P • ~Q'.

Note that this formula is in DNF.

Also, note that it is equivalent to a different statement, '~(P ∨ Q)', by DeMorgan's Law.

There are multiple formulas which will yield the same truth table.

In fact, there are infinitely many ways to produce each truth table.

(For example, one can always just add pairs of tildes to a formula.)

Case 3:

For each row in which the statement is true, perform the method described in Case 2.
Then, form the disjunction of all the resulting formulas.

Example:
Consider a formula with three variables.
P=TTTTFFFF; Q=TTFFTTFF; R=TFTFTFTF
Major operator=TFFTFFFF
To construct a formula with that truth table, consider only the first and fourth rows.
$(P \bullet Q \bullet R) \vee (P \bullet {\sim}Q \bullet {\sim}R)$
(Punctuation can easily be added to make the formula well-formed.)

QED

X. Smaller sets of adequate connectives

Theorem 5: The set {∨, ~} is adequate.

Proof 5:

By Theorem 4, we can write a formula for any truth table using as connectives only those in the set {∨, •, ~}.
'P • Q' is equivalent to '~(~P ∨ ~Q)'.
So, we can replace any occurrence of '•' in any formula, according to the above equivalence.

QED

Theorem 6: The set {•, ~} is adequate.
Theorem 7: The set {~, ⊃} is adequate.

XI. Inadequate sets

Theorem 8: The set {⊃, ∨} is inadequate.

Proof 8:

Both 'P ⊃ Q' and 'P ∨ Q' are true when P and Q are both true.
Thus, using these connectives we can never construct a truth table with a false first row.

QED

Theorem 9: The set {⊃} is inadequate.

Proof 9:

Consider the truth table for conjunction: TFFF.
We want to construct a formula, using ⊃ as the only connective, which yields the same truth table.
Imagine that we have such a formula, and imagine the smallest such formula.
Since, the only way to get an F with ⊃ is with a false consequent, the truth table of the consequent of our formula must either be TFFF or FFFF.
Since we are imagining that our formula is the smallest formula which yields TFFF, the consequent of our formula must be a contradiction.
But, the only way to get a contradiction, using ⊃ alone, is to have one already!
Since we can not construct the contradiction, we can not construct the conjunction.

QED

Theorem 10: The set {~} is inadequate.
Proof 10:
The only possible truth tables with one variable and ~ are TF and FT.
Thus, we can not generate TT or FF.
QED

XI. The Sheffer stroke, '|'

p	|	q
T	F	T
T	T	F
F	T	T
F	T	F

Theorem 11: The set {|} is adequate.
Proof 11:
'~P' is logically equivalent to 'P | P'.
'P • Q' is logically equivalent to '(P | Q) | (P | Q)'.
By Theorem 6, {~, •} is adequate.
QED

XII. the Peirce arrow, '↓', also called joint denial, or neither-nor.

p	↓	q
T	F	T
T	F	F
F	F	T
F	T	F

Theorem 12: The set {↓} is adequate.
Proof 12:
'~P' is equivalent to 'P ↓ P'.
'P ∨ Q' is equivalent to '(P ↓ Q) ↓ (P ↓ Q)'.
Theorem 5.
QED

XIII. The limit of adequacy

Theorem 13: ↓ and | are the only connectives which are adequate by themselves.
Proof 13:

Imagine we had another adequate connective, #.
We know the first rows must be false, by the reasoning in Proof 8.
Similar reasoning fills in the last row.

p	#	q
T	F	T
T		F
F		T
F	T	F

Thus, '~P' is equivalent to 'P # P'.
Now, we need to fill in the other rows.
If the remaining two rows are TT, then we have '|'.
If the remaining two rows are FF, then we have '↓'.
So, the only other possibilities are TF and FT.
FT yields FFTT, which is just '~P'.
TF yields FTFT, which is just '~Q'.
By Theorem 10, {~} is inadequate.

QED

XIV. For further reading/papers:

Geoffrey Hunter, *Metalogic*. The results above are mostly contained in §21. The references below are mostly found there, as well. His notation is a bit less friendly, but the book is wonderful, and could be the source of lots of papers.

Elliott Mendelson, *Introduction to Mathematical Logic*. Mendelson discusses adequacy in §1.3. His notation is less friendly than Hunter's, but the exercises lead you through some powerful results.

Emil Post, "Introduction to a General Theory of Elementary Propositions", reprinted in van Heijenhoort. The notation is different, but the concepts are not too difficult. It would be interesting to translate into a current notation, and present some of the results.

Several papers from C.S. Peirce initially explored the single adequate connectives. They might be fun to work through. I can give you references.

While there are no other adequate connectives, there are other connectives. You might be able to work up a paper considering some of those.

Also, you could think about why there are only unary and binary connectives.