Philosophy 240: Symbolic Logic
Fall 2008
Mondays, Wednesdays, Fridays: 9am - 9:50am

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Propositions and Logical Truths Handout

A. If it is raining, then I will be unhappy.
B. If it is raining, then I will get wet.
C. If it is raining, then it is raining.

D: The law of the excluded middle: $\alpha \vee \sim\alpha$.

E. The cat dances. The cat dances.
F. Mississippi

G. Disjunctive syllogism:

$\alpha \vee \beta$
$\sim\alpha \quad / \beta$

H. The cat either dances or sings.
She doesn't dance.
Therefore, she sings.

I. Either the cat dances or the cat sings.
It is not the case that the cat dances.
Therefore, the cat sings.

J. El gato baila.
K. The cat dances.

L. Propositions: the meanings of sentence types
- abstract objects
- mind-independent
- indicated by 'that': 'the cat dances' and 'el gato baila' both express the proposition that the cat dances
- language-independent

M. 'Visiting relatives can be annoying'.

N: The law of non-contradiction (or the law of contradiction): $\sim(\alpha \bullet \sim\alpha)$
O: $\alpha \supset (\beta \supset \alpha)$
P: $[\alpha \supset (\beta \supset \gamma)] \supset [(\alpha \supset \beta) \supset (\alpha \supset \gamma)]$

Axiom systems

Ax_1: $\alpha \supset (\beta \supset \alpha)$
Ax_2: $[\alpha \supset (\beta \supset \gamma)] \supset [(\alpha \supset \beta) \supset (\alpha \supset \gamma)]$
Ax_3: $(\sim\alpha \supset \sim \beta) \supset (\beta \supset \alpha)$
Rule of substitution: Any wff which results from consistently substituting wffs for each of the terms in any of the axioms above is a theorem.
Rule of inference: Modus Ponens

A proof of '$P \supset P$'

1. $P \supset [(P \supset P) \supset P]$	Ax_1
2. $\{P \supset [(P \supset P) \supset P]\} \supset \{[P \supset (P \supset P)] \supset (P \supset P)\}$	Ax_2
3. $[P \supset (P \supset P)] \supset (P \supset P)$	1, 2, MP
4. $P \supset (P \supset P)$	Ax_1
5. $P \supset P$	3, 4, MP

QED

A proof of '$(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$'

1. $[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]$	Ax_2
2. $\{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\} \supset \{(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}\}$	Ax_1
3. $(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}$	1, 2, MP
4. $\{(Q \supset R) \supset \{[P \supset (Q \supset R)] \supset [(P \supset Q) \supset (P \supset R)]\}\} \supset$ $\{\{(Q \supset R) \supset [P \supset (Q \supset R)]\} \supset \{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\}\}$	Ax_2
5. $\{(Q \supset R) \supset [P \supset (Q \supset R)]\} \supset \{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\}$	4, 3, MP
6. $(Q \supset R) \supset [P \supset (Q \supset R)]$	Ax_1
7. $(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]$	5, 6, MP
8. $\{(Q \supset R) \supset [(P \supset Q) \supset (P \supset R)]\} \supset$ $\{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}$	Ax_2
9. $[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]$	8, 7, MP
10. $\{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\} \supset$ $\{(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}\}$	Ax_1
11. $(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}$	10, 9, MP
12. $\{(P \supset Q) \supset \{[(Q \supset R) \supset (P \supset Q)] \supset [(Q \supset R) \supset (P \supset R)]\}\} \supset$ $\{\{(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]\} \supset \{(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]\}\}$	Ax_2
13. $\{(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]\} \supset \{(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]\}$	12, 11, MP
14. $(P \supset Q) \supset [(Q \supset R) \supset (P \supset Q)]$	Ax_1
15. $(P \supset Q) \supset [(Q \supset R) \supset (P \supset R)]$	13, 14, MP

QED

Exercises:

1. Prove '$(\sim P \supset P) \supset P$' (It can be done in about 11 lines.)
2. Prove '$[P \supset (Q \supset R)] \supset [Q \supset (P \supset R)]$'

Just About All Of Mathematics

Propositional Logic (PL), following Mendelson, *Introduction to Mathematical Logic*
The symbols are ~, ⊃, (,), and the statement letters A_i, for all positive integers i.

All statement letters are wffs.

If 'α' and 'β' are wffs, so are '~α' and '(α ⊃ β)'.

If α, β, and γ are wffs, then the following are axioms:

A1: $(\alpha \supset (\beta \supset \alpha))$

A2: $((\alpha \supset (\beta \supset \gamma)) \supset ((\alpha \supset \beta) \supset (\alpha \supset \gamma)))$

A3: $((\sim\beta \supset \sim\alpha) \supset ((\sim \beta \supset \alpha) \supset \beta))$

There is one rule of inference:

β is a direct consequence of α and (α ⊃ β)

First-Order Predicate Logic with Equality (QL=), again following Mendelson, with adjustments
The language is that of PL, with the addition of:

the universal quantifier, ∀

individual variables, x_i, for all positive integers i

individual constants, a_i, for all positive integers i

predicate letters, A^n_k, for all positive integers n and k

function letters, f^n_k, for all positive integers n and k

Note: the superscripts on the predicate and function letters indicate the number of argument places they take; subscripts serve just to distinguish among different predicates and functions.

Variables, constants, and functions of terms with the appropriate number of argument places are called terms:

Wffs:

1. Any predicate letter applied to the appropriate collection of terms (according to its number of argument spaces) is a wff.
2. If α and β are wffs, and x is a variable so are:

   ~α

   (α ⊃ β), and

   ∀x α

Axioms:

A1 - A3, from PL are axioms.

A4: $\forall x_i\ \alpha(x_i) \supset \alpha(t)$, where $\alpha(x_i)$ and $\alpha(t)$ are wffs involving x_i and a term t, respectively[1]

A5: $\forall x_i\ (\alpha \supset \beta) \supset (\alpha \supset \forall x_i\ \beta)$

A6: $\forall x_i\ A^2_1(x_i, x_i)$

A7: $A^2_1(x_i, x_j) \supset (\alpha(x_i) \supset \alpha(x_j))$

Note: A^2_1 is the special identity predicate.

We abbreviate A6 as '$\forall x_i\ x_i{=}x_i$', and A7 as '$x_i = x_j \supset (\alpha(x_i) \supset \alpha(x_j))$'

Rules of Inference:

β is a direct consequence of α and (α ⊃ β)

$\forall x_i$ α is a direct consequence of α

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[1] I omit for brevity, here, and in A5, essential restrictions involving the binding of variables.

Zermelo-Fraenkel Set Theory (ZF), again following Mendelson, but with adjustments
ZF may be written in the language and deductive system of QL, with a special predicate letter, $\in$, added.
The axioms are:

Substitutivity: $(x)(y)(z)[y=z \supset (y\in x \equiv z\in x)]$
Pairing: $(x)(y)(\exists z)(u)[u\in z \equiv (u = x \vee u = y)]$
Null Set: $(\exists x)(y){\sim}x\in y$
Note: the null set axiom ensures the existence of an empty set, so we can introduce a constant, $\varnothing$, such that $(x){\sim}x\in\varnothing$.
Sum Set: $(x)(\exists y)(z)[z\in y \equiv (\exists v)(z\in v \bullet v\in x)]$
Power Set: $(x)(\exists y)(z)[z\in y \equiv (u)(u\in z \supset u\in x)]$
Selection: $(x)(\exists y)(z)[z\in y \equiv (z\in x \bullet \mathscr{F}u)]$, for any formula $\mathscr{F}$ not containing y as a free variable.
Infinity: $(\exists x)(\varnothing\in x \bullet (y)(y\in x \supset Sy\in x)$
Note: 'Sy' stands for $y\cup\{y\}$, the definitions for the components of which are standard.

Peano Arithmetic (PA)
PA may be written in the language of QL, with special constants and function letters for the numbers and functions, like multiplication.
Axioms

P1: 0 is a number
P2: The successor, x', of every number is a number
P3: 0 is not the successor of any number
P4: If x'=y' then x=y
P5: If P is a property that may (or may not) hold for any number, and if
i. 0 has P; and
ii. for any x, if x has P then x' has P;
then all numbers have P.

Note: P5 is also called mathematical induction, and is actually a schema of an infinite number of axioms.
The axioms of PA are constructible out of the axioms of ZF, and other set theories.
With set theory, we can also construct the theories of the reals, rationals, and complex numbers.

Birkhoff's Postulates for Geometry, following James Smart, *Modern Geometries*

Postulate I: Postulate of Line Measure. The points A, B,... of any line can be put into a 1:1 correspondence with the real numbers x so that $|x_B-x_A| = d(A,B)$ for all points A and B.

Postulate II: Point-Line Postulate. One and only one straight line l contains two given distinct points P and Q.

Postulate III: Postulate of Angle Measure. The half-lines l, m... through any point O can be put into 1:1 correspondence with the real numbers a(mod 2π) so that if $A\neq0$ and $B\neq0$ are points on l and m, respectively, the difference $a_m - a_l \pmod{2\pi}$ is angle$\triangle$AOB. Further, if the point B on m varies continuously in a line r not containing the vertex O, the number a_m varies continuously also.

Postulate IV: Postulate of Similarity. If in two triangles $\triangle$ABC and $\triangle$A'B'C', and for some constant k>0, d(A', B') = kd(A, B), d(A', C')=kd(A, C) and $\triangle$B'A'C'=$\pm\triangle$BAC, then d(B', C')=kd(B,C), $\triangle$C'B'A'=$\pm\triangle$CBA, and $\triangle$A'C'B'=$\pm\triangle$ACB.

As with PA, Birkhoff's postulates can be constructed set-theoretically.
Other mathematical theories are constructed mainly by adding definitions.