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Rules of Replacement, II: §7.4

I. The Last Five Rules of Replacement
See appendix for truth tables proving the equivalence for each.

Transposition (Trans)
P ⊃ Q :: ~Q ⊃ ~P
You may switch the antecedent and consequent of a conditional statement, as long as you negate (or un-negate) both.
Often used with (HS).
Also, traditionally, called the 'contrapositive'.

Sample Derivation:
1. A ⊃ B
2. D ⊃ ~B / A ⊃ ~D
3. ~~B ⊃ ~D 2, Trans
4. A ⊃ ~D 1, 3, DN, HS
QED

This can be tricky when only one term is negated:
A ⊃ ~B becomes, by Trans:
~~B ⊃ ~A which becomes, by DN
B ⊃ ~A
Equivalently, but doing the double negation first:
A ⊃ ~B becomes, by DN:
~~A ⊃ ~B becomes, by Trans:
B ⊃ ~A
Either way, you can include the DN on the line with Trans.

Material Implication (Impl)
P ⊃ Q :: ~P ∨ Q
Changing a statement from a disjunction to a conditional, or vice versa.
It's often easier to work with disjunctions.
You can use (DM) to get conjunctions.
You may be able to use distribution, which doesn't apply to conditionals.
On the other hand, sometimes, you just want to work with conditionals.
You can use (HS) and (MP).
Proofs are overdetermined by our system - there are many ways to do them.

Sample Derivation:
1. G ⊃ ~E
2. E ∨ F / G ⊃ F
3. ~~E ∨ F 2, DN
4. ~E ⊃ F 3, Impl
5. G ⊃ F 1, 4, HS
QED

Material Equivalence (Equiv)

$P \equiv Q :: (P \supset Q) \cdot (Q \supset P)$
$P \equiv Q :: (P \cdot Q) \vee (\sim P \cdot \sim Q)$
(Almost) the only thing you can do with a biconditional.
Two distinct versions.
If you have a biconditional in your premises, you can unpack it in either way.
If you need one in your conclusion, you can get the pieces and then use this rule.
This is easier with the first definition.
Just get $P \supset Q$
Then get $Q \supset P$
Then use (Conj).

Sample Derivation
1. $\sim[(K \supset \sim H) \cdot (\sim H \supset K)]$
2. $(I \cdot J) \supset (K \equiv \sim H)$ / $\sim(I \cdot J)$
3. $\sim(K \equiv \sim H)$ 1, Equiv
4. $\sim(I \cdot J)$ 2, 3, MT
QED

Exportation (Exp)
$P \supset (Q \supset R) :: (P \cdot Q) \supset R$
You can can get to (MP) or (MT) sometimes with it.

Sample Derivation:
1. $L \supset (M \supset N)$
2. $\sim N$ / $\sim L \vee \sim M$
3. $(L \cdot M) \supset N$ 1, Exp
4. $\sim(L \cdot M)$ 3, 2, MT
5. $\sim L \vee \sim M$ 4, DM
QED

Tautology (Taut)
$P :: P \cdot P$
$P :: P \vee P$
Eliminates redundancy.

Sample Derivation:
1. $O \supset \sim O$ / $\sim O$
2. $\sim O \vee \sim O$ 1, Impl
3. $\sim O$ 2, Taut
QED

II. Some more potentially helpful examples.
Some of these may be useful as elements of other, longer proofs.
Others contain useful tricks which may come in handy in other proofs.

1)
1. $\sim A$ / $A \supset B$
2. $\sim A \vee B$ 1, Add
3. $A \supset B$ 2, Impl
QED

2)
1. E / $F \supset E$
2. $\sim F \vee E$ 1, Add, Com

3. F ⊃ E	2, Impl

QED

3)

1. G ⊃ (H ⊃ I)	/ H ⊃ (G ⊃ I)
2. (G · H) ⊃ I	1, Exp
3. (H · G) ⊃ I	2, Com
4. H ⊃ (G ⊃ I)	3, Exp

QED

4)

1. O ⊃ (P · Q)	/ O ⊃ P
2. ~O ∨ (P · Q)	1, Impl
3. (~O ∨ P) · (~O ∨ Q)	2, Dist
4. ~O ∨ P	3, Simp
5. O ⊃ P	4, Impl

QED

5)

1. (R ∨ S) ⊃ T	/ R ⊃ T
2. ~(R ∨ S) ∨ T	1, Impl
3. (~R · ~S) ∨ T	2, DM
4. T ∨ (~R · ~S)	3, Com
5. (T ∨ ~R) · (T ∨ ~S)	4, Dist
6. T ∨ ~R	5, Simp
7. ~R ∨ T	6, Com
8. R ⊃ T	7, Impl

QED

6)

1. W ⊃ X	
2. Y ⊃ X	/ (W ∨ Y) ⊃ X
3. (W ⊃ X) · (Y ⊃ X)	1, 2, Conj
4. (~W ∨ X) · (~Y ∨ X)	3, Impl, Impl
5. (X ∨ ~W) · (X ∨ ~Y)	4, Com, Com
6. X ∨ (~W · ~Y)	5, Dist
7. (~W · ~Y) ∨ X	6, Com
8. ~(W ∨ Y) ∨ X	7, DM
9. (W ∨ Y) ⊃ X	8, Impl

QED

7)

1. (J ∨ K) ⊃ (L · M)	
2. ~J ⊃ (N ⊃ ~N)	
3. ~L	/ ~N
4. ~L ∨ ~M	3, Add
5. ~(L · M)	4, DM
6. ~(J ∨ K)	1, 5, MT
7. ~J ·~K	6, DM
8. ~J	7, Simp
9. N ⊃ ~N	2, 8, MP
10. ~N ∨ ~N	9, Impl
11. ~N	10, Taut

QED

III. Exercises. Derive the conclusions of each of the following arguments using the Rules of Inference and Replacement.

1)
1. (O · P) ⊃ Q
2. O / P ⊃ Q

2)
1. R ⊃ (S · ~T) / ~R ∨ ~T

3)
1. U ≡ W
2. W / U

4)
1. (H ∨ I) ⊃ [J · (K · L)]
2. I / J · K

5)
1. (L · M) ⊃ N
2. (L ⊃ N) ⊃ O / M ⊃ O

6)
1. A · (B ∨ F)
2. A ⊃ [B ⊃ (D · E)]
3. (A · F) ⊃ ~(D ∨ E) / D ≡ E

IV. Three challenging derivations. Try them.

1)
1. A ⊃ B
2. B ⊃ D
3. D ⊃ A
4. A ⊃ ~D / ~A · ~D

2)
1. (I · E) ⊃ ~F
2. F ∨ (G · H)
3. I ≡ E / I ⊃ G

3)
1. (J ⊃ J) ⊃ (K ⊃ K)
2. (K ⊃ L) ⊃ (J ⊃ J) / K ⊃ K

V. Appendix: Proofs of the Logical Equivalence of the Last Five Rules of Replacement

Transposition: P ⊃ Q :: ~Q ⊃ ~P

P	⊃	Q
T	**T**	T
T	**F**	F
F	**T**	T
F	**T**	F

~	Q	⊃	~	P
F	T	**T**	F	T
T	F	**F**	F	T
F	T	**T**	T	F
T	F	**T**	T	F

Material Implication: P ⊃ Q :: ~P ∨ Q

P	⊃	Q
T	**T**	T
T	**F**	F
F	**T**	T
F	**T**	F

~	P	∨	Q
F	T	**T**	T
F	T	**F**	F
T	F	**T**	T
T	F	**T**	F

Material Equivalence: P ≡ Q :: (P ⊃ Q) · (Q ⊃ P)

P	≡	Q
T	**T**	T
T	**F**	F
F	**F**	T
F	**T**	F

(P	⊃	Q)	·	(Q	⊃	P)
T	T	T	**T**	T	T	T
T	F	F	**F**	F	T	T
F	T	T	**F**	T	F	F
F	T	F	**T**	F	T	F

Material Equivalence: P ≡ Q :: (P · Q) ∨ (~P · ~Q)

P	≡	Q
T	**T**	T
T	**F**	F
F	**F**	T
F	**T**	F

(P	·	Q)	∨	(~	P	·	~	Q)
T	T	T	**T**	F	T	F	F	T
T	F	F	**F**	F	T	F	T	F
F	F	T	**F**	T	F	F	F	T
F	F	F	**T**	T	F	T	T	F

Exportation: (P · Q) ⊃ R :: P ⊃ (Q ⊃ R)

(P	·	Q)	⊃	R
T	T	T	**T**	T
T	T	T	**F**	F
T	F	F	**T**	T
T	F	F	**T**	F
F	F	T	**T**	T
F	F	T	**T**	F
F	F	F	**T**	T
F	F	F	**T**	F

P	⊃	(Q	⊃	R)
T	**T**	T	T	T
T	**F**	T	F	F
T	**T**	F	T	T
T	**T**	F	T	F
F	**T**	T	T	T
F	**T**	T	F	F
F	**T**	F	T	T
F	**T**	F	T	F

Tautology: P :: P ∨ P

P
T
F

P	∨	P
T	**T**	T
F	**F**	F

Tautology: P :: P · P

P
T
F

P	·	P
T	**T**	T
F	**F**	F