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Logical Truths, §7.7

I. What is a logical truth?

A logical truth is a statement that requires no justification outside of logic.

Tautologies are theorems of any system of propositional logic.
Laws of propositional logic are all tautologies.
They can be derived regardless of the premises, or without any premises.

Some systems have a rule by which you can enter laws of logic into any line in a proof
But then we still need an independent method of determining that a statement is a theorem.
Truth tables suffice for this.
Some systems take a few logical truths as axioms, and reduce the rules of inference, even to just one rule, normally (MP).

Some important theorems:
P ∨ ~P is called the law of the excluded middle
~(P · ~P) is called the law of non-contradiction (or sometimes, ironically, the law of contradiction).
There are infinitely many theorems.

A couple of other examples of logical truths of propositional logic:
P ⊃ (Q ⊃ P)
[P ⊃ (Q ⊃ R)] ⊃ [(P ⊃ Q) ⊃ (P ⊃ R)]

II. Sample derivations of logical truths.

We can use either conditional proof or indirect proof to derive logical truths.
If the main connective is a conditional or a biconditional, we generally use conditional proof.
If the main connective isa disjunction or a negation, we generally use indirect proof.
If the main connective is a conjunction, we look to the main connectives of each conjunct to determine the best method of proof.

Using CP:
Show that ‘(P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]’ is a logical truth.

*1. P ⊃ Q	ACP
* *2. Q ⊃ R	ACP
* *3. P ⊃ R	1, 2, HS
*4. (Q ⊃ R) ⊃ (P ⊃ R)	2-3, CP
5. (P ⊃ Q) ⊃ [(Q ⊃ R) ⊃ (P ⊃ R)]	1-4, CP

QED

Using IP:
Show that ‘(P ⊃ Q) ∨ (~Q ⊃ P)’ is a logical truth.

*1. ~[(P ⊃ Q) ∨ (~Q ⊃ P)]	AIP
*2. ~(P ⊃ Q) · ~(~Q ⊃ P)	1, DM
*3. ~(P ⊃ Q)	2, Simp
*4. ~(~P ∨ Q)	3, Impl
*5. P · ~Q	4, DM, DN

*6. ~(~Q ⊃ P)	2, Com, Simp	
*7. ~(Q ∨ P)	6, Impl, DN	
*8. ~Q · ~P	7, DM	
*9. P	5, Simp	
*10. ~P	8, Com, Simp	
*11. P · ~P	9, 10, Conj	
12. (P ⊃ Q) ∨ (~Q ⊃ P)	1-11, IP	

QED

Sometimes, we might need a logical truth as an intermediate step in a proof:

1. B ⊃ [(D ⊃ D) ⊃ E]		
2. E ⊃ {[F ⊃ (G ⊃ F)] ⊃ (H · ~H)} / ~B		
*3. B	AIP	
*4. (D ⊃ D) ⊃ E	1, 3, MP	Note that 'D ⊃ D' is derivable using IP or CP
* *5. ~(D ⊃ D)	AIP	
* *6. ~(~D ∨ D)	5, Impl	
* *7. D · ~D	6, DM, DN	
*8. D ⊃ D	5-7, IP, DN	
*9. E	4, 8, MP	
*10. [F ⊃ (G ⊃ F)] ⊃ (H · ~H)	2, 9, MP	Note the antecedent here is another logical truth
* *11. F	ACP	
* *12. F ∨ ~G	11, Add	
* *13. ~G ∨ F	12, Com	
* *14. G ⊃ F	13, Impl	
*15. F ⊃ (G ⊃ F)	11-14, CP	
*16. H · ~H	10, 15, MP	
17. ~B	3-16, IP	

QED

III. Exercises. Derive each of the following logical truths, using CP or IP.

1) [(A ⊃ B) · A] ⊃ B
2) (P ∨ P) ⊃ P
3) (A ⊃ B) ⊃ [A ⊃ (A · B)]
4) (A ⊃ B) ∨ (~A ⊃ C)
5) (P ⊃ Q) ⊃ {(P ⊃ R) ⊃ [P ⊃ (Q · R)]}