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Identity theory, §8.7

I. The identity predicate is a special predicate, with a special logic.

Consider the following logical derivation:

Superman can fly.	Fs
Superman is Clark Kent.	???
So, Clark Kent can fly.	Fc

We know the two people are the same, so anything true of one, is true of the other.  
 This is called the Law of the Indiscernibility of identicals:  $(x)(y)[(x=y) \supset (\mathcal{F}x \equiv \mathcal{F}y)]$   
 ‘=’ is just a special predicate, which we could write as ‘Exy’  
 Instead, we introduce a new symbol, ‘=’

The three rules governing Identity (ID)

1)  
 $a=a$   
 For any constant ‘a’, ‘a’ is identical to itself.  
 We can add, as a line in any proof, a statement of this form.  
 This only applies to constants.

2)  
 $a=b :: b=a$   
 Identity is commutative  
 This can be applied to variables, as well as to constants.

3)  
 $\mathcal{F}a$   
 $a=b \quad / \mathcal{F}b$   
 If  $a=b$ , then, you may rewrite any formula containing ‘a’ with ‘b’ in the place of ‘a’ throughout.

II. Translation with identity

Simple ID  
 Clark Kent is Superman  $c=s$   
 Mary Ann Evans is George Eliot  $m=g$

Only and except:  
 John loves Mary  $Ljm$   
 John only loves Mary:  $Ljm \cdot (x)(Ljx \supset x=m)$   
 Only John loves Mary:  $Ljm \cdot (x)(Lxm \supset x=j)$   
 Everyone loves Mary  $(x)(Px \supset Lxm)$   
 Everyone except John loves Mary:  $\sim Ljm \cdot (x)[(Px \cdot x \neq j) \supset Lxm]$   
 Note: ‘ $x \neq j$ ’ is just shorthand for ‘ $\sim x=j$ ’. Negation applies to the identity predicate, and not to the objects.

Superlatives:  
 Degas is a better impressionist than Monet:  $Id \cdot Im \cdot Bdm$   
 Degas is the best impressionist:  $Id \cdot (x)[(Ix \cdot x \neq d) \supset Bdx]$   
 Bill Gates is the geek with the most money  $Gg \cdot (x)[(Gx \cdot x \neq g) \supset Mgx]$   
 Philadelphia is the nearest major city  $Mp \cdot (x)[(Mx \cdot x \neq p) \supset Npx]$

Numerical operators:

There is only one applicant for the job:  $(\exists x)[Ax \cdot (y)(Ay \supset x=y)]$   
 There is at least one applicant for the job:  $(\exists x)Ax$   
 There are at least two applicants:  $(\exists x)(\exists y)[(Ax \cdot Ay) \cdot x \neq y]$   
 There are exactly two applicants:  $(\exists x)(\exists y)\{[(Ax \cdot Ay) \cdot x \neq y] \cdot (z)[Az \supset (z=x \vee z=y)]\}$   
 There is at most one applicant:  $(x)(y)[(Ax \cdot Ay) \supset x=y]$   
 There are at most two applicants:  $(x)(y)(z)[(Ax \cdot Ay \cdot Az) \supset (x=y \vee x=z \vee y=z)]$   
 Note: this statement makes no existential commitments!

### III. Definite descriptions.

Consider the sentence 'The King of France is bald'

We might translate it as 'Bk'.

'Bk' is false, since there is no King of France.

So, '~Bk' should be true, since it's the negation of a false statement.

That means that 'It's false that the King of France is bald' is true.

Which seems to imply that the King of France has hair.

In fact, we want both 'The King of France is bald' and 'The King of France is not bald' to be false.

So, we had better translate the sentence differently.

'The King of France' is a definite description

It refers to one specific object without using a name.

There are two ways to refer: by name or by description (E.g. the person who, the thing that)

Bertrand Russell's analysis for definite descriptions:

They're both false, due to a false presupposition.

They're really complex statements, involving a definite description.

'The King of France is bald' entails three simpler expressions:

A. There is a King of France.

B. There is only one King of France.

C. That thing is bald.

So, the proposition is false because clause A is false.

'The King of France is not bald' is also false, for the same reason

The negation only affects the third clause.

The first is still the same, and still false.

Another example: The country called a sub-continent is India.

1. There is a country called a sub-continent

2. There is only one such country

3. That country is identical with India

So, we translate as:  $(\exists x)\{(Cx \cdot Sx) \cdot (y)[(Cy \cdot Sy) \supset y=x] \cdot x=I\}$

Another example: 'The author of Waverly was a genius' becomes  $(\exists x)\{Wx \cdot (y)[Wy \supset y=x] \cdot Gx\}$

### IV. More examples.

1. Everything is identical with itself.

$(x)x=x$

2. Nothing is distinct from itself.

$(x)\sim \sim x=x$

3. The discoverer of Polonium is Polish. (Dx, Px)

$(\exists x)\{Dx \cdot (y)(Dy \supset y=x) \cdot Px\}$

4. At most two persons invented the airplane. (Px, Ix)

$(x)(y)(z)[(Px \cdot Ix \cdot Py \cdot Iy \cdot Pz \cdot Iz) \supset (x=y \vee x=z \vee y=z)]$

5. There is exactly one dollar bill in my wallet. (Dx, Wx)

$(\exists x)\{(Dx \cdot Wx) \cdot (y)[(Dy \cdot Wy) \supset y=x]\}$

6. Adriana is a better dancer than Rene. (a, r, Dx, Bxy: x is better than y)

Da · Dr · Bar

7. Adriana is the best dancer.

Da · (x)[(Dx · ~x=a) ⊃ Bax]

8. There are exactly two dancers better than Adriana.

(∃x)(∃y){(Dx · Bxa · Dy · Bya · ~x=y) · (z)[(Dz · Bza) ⊃ (z=x ∨ z=y)]}

V. Exercises A. Translate the following:

1. Everything is identical with something.

2. There are prime numbers. (Px, Nx)

3. Two is the only even prime number. (t, Ex, Px, Nx)

4. There is exactly one even prime number.

5. There are at least two odd prime numbers. (Ox, Px, Nx)

6. All prime numbers are odd except the number two.

7. The murderer was Colonel Mustard. (m, Mx)

8. There is at least one dancer better than Rene.

9. Rene is the worst dancer.

10. Goliath is the tallest human. (g, Hx, Txy: x is taller than y)

VI. Derivations

Consider the original problem:

Superman can fly.

Superman is Clark Kent

∴ Clark Kent can fly.

1. Fs

2. s=c / Fc

3. Fc 1, 2, ID

QED

Using the commutative Id rule:

1. a=b ⊃ j=k

2. b=a

3. Fj / Fk

4. a=b 2, Id

5. j=k 1, 4, MP

6. Fk 3, 5, Id

QED

To derive the negation of an identity statement, you must use (IP):

1. Rm

2. ~Rj / m≠j

\*3. m=j

\*4. Rj

\*5. Rj · ~Rj

6. m≠j

QED

Using the reflexive Id rule:

1. (x)(~Gx ⊃ x≠d) / Gd

\*2. ~Gd AIP

\*3. ~Gd ⊃ d≠d 1, UI

\*4. d=d Id

\*5. d≠d 3, 2, MP

\*6. d=d · d≠d 4, 5, Conj

7. Gd  
QED

Proof with existential conclusion:

1. Rab
  2.  $(\exists x)\sim Rxb$  /  $(\exists x)\sim x=a$
  3.  $\sim Rcb$  2, EI
  - \*4.  $c=a$  AIP
  - \*5. Rcb 1, Id
  - \*6.  $Rcb \cdot \sim Rcb$  5, 3, Conj
  7.  $\sim c=a$  4-6, IP
  8.  $(\exists x)\sim x=a$  7, EG
- QED

A final example: The Faulkner scholar at Swarthmore is very learned. Therefore, all Faulkner scholars at Swarthmore are very learned.

1.  $(\exists x)\{[(Sx \cdot Fx) \cdot (y)[(Sy \cdot Fy) \supset x=y]] \cdot Lx\}$  /  $(x)[(Sx \cdot Fx) \supset Lx]$
  - \*2.  $\sim(x)[(Sx \cdot Fx) \supset Lx]$  AIP
  - \*3.  $(\exists x)\sim[(Sx \cdot Fx) \supset Lx]$  2, CQ
  - \*4.  $\sim[(Sa \cdot Fa) \supset La]$  3, EI
  - \*5.  $\sim[\sim(Sa \cdot Fa) \vee La]$  4, Impl
  - \*6.  $(Sa \cdot Fa) \cdot \sim La$  5, DM, DN
  - \*7.  $\{(Sb \cdot Fb) \cdot (y)[(Sy \cdot Fy) \supset b=y]\} \cdot Lb$  1, EI (to b)
  - \*8.  $(y)[(Sy \cdot Fy) \supset b=y]$  7, Simp, Com, Simp
  - \*9.  $(Sa \cdot Fa) \supset b=a$  8, UI (to a)
  - \*10.  $Sa \cdot Fa$  6, Simp
  - \*11.  $b=a$  9, 10, MP
  - \*12. Lb 7, Simp
  - \*13. La 12, 11, Id
  - \*14.  $\sim La$  6, Com, Simp
  - \*15.  $La \cdot \sim La$  13, 14, Conj
  16.  $(x)[(Sx \cdot Fx) \supset Lx]$  2-15, IP
- QED

VI. Exercises B. Derive the conclusions of each of the following arguments.

- 1)
1.  $(x)(Dx \supset Ex)$
  2. Da
  3. a=b / Eb

- 2)
1.  $(x)(Ax \supset Bx)$
  2.  $\sim Bf$
  3. Ae / f≠e

- 3)
1.  $(x)(Hx \supset Jx)$
  2.  $(x)(Kx \supset Lx)$
  3. Hd · Kc
  4. c=d / Jc · Ld

- 4)
1.  $(x)(y)(x=y)$
  2.  $(x)Mxx$  / Mab

5)

1.  $(x)[(\exists y)Kxy \supset (\exists z)Kzx]$
2.  $(\exists x)(Kxg \cdot x=b)$  /  $(\exists z)Kzb$

6)

1.  $(\exists x)Hx$
2.  $(x)(y)[(Hx \cdot Hy) \supset x=y]$  /  $(\exists x)[Hx \cdot (y)(Hy \supset x=y)]$

## VII. Solutions.

### Answers to Exercises A.

1.  $(x)(\exists y)x=y$
  2.  $(\exists x)(Px \cdot Nx)$
  3.  $Et \cdot Pt \cdot Nt \cdot (x)[(Ex \cdot Px \cdot Nx) \supset x=t]$
  4.  $(\exists x)\{(Ex \cdot Px \cdot Nx) \cdot (y)[(Ey \cdot Py \cdot Ny) \supset y=x]\}$
  5.  $(\exists x)(\exists y)(Ox \cdot Px \cdot Nx \cdot Oy \cdot Py \cdot Ny \cdot \sim x=y)$
  6.  $(x)[(Px \cdot Nx \cdot \sim x=t) \supset Ox]$
  7.  $(\exists x)[Mx \cdot (y)(My \supset y=x) \cdot x=m]$
  8.  $Dr \cdot (\exists x)(Dx \cdot Bxr)$
  9.  $Dr \cdot (x)[(Dx \cdot \sim x=r) \supset Bxr]$
  10.  $Hg \cdot (x)[(Hx \cdot \sim x=g) \supset Tgx]$
- Or, using definite descriptions:  $(\exists x)\{(y)[(Hy \cdot \sim y=x) \supset Txy] \cdot (z)\{(y)[(Hy \cdot \sim y=z) \supset Tzy] \supset z=x\} \cdot x=g\}$

### A solution to Exercise B.6:

1.  $(\exists x)Hx$
2.  $(x)(y)[(Hx \cdot Hy) \supset x=y]$  /  $(\exists x)[Hx \cdot (y)(Hy \supset x=y)]$
3.  $Ha$ 
  - \*4.  $\sim(\exists x)[Hx \cdot (y)(Hy \supset x=y)]$  1, EI
  - \*5.  $(x)\sim[Hx \cdot (y)(Hy \supset x=y)]$  AIP
  - \*6.  $(x)[\sim Hx \vee \sim(y)(Hy \supset x=y)]$  4, CQ
  - \*7.  $\sim Ha \vee \sim(y)(Hy \supset a=y)$  5, DM
  - \*8.  $\sim(y)(Hy \supset a=y)$  6, UI
  - \*9.  $(\exists y) \sim(Hy \supset a=y)$  7, 3, DN, DS
  - \*10.  $\sim(Hb \supset a=b)$  8, CQ
  - \*11.  $\sim(\sim Hb \vee a=b)$  9, EI
  - \*12.  $Hb \cdot \sim a=b$  10, Impl
  - \*13.  $Hb$  11, DM, DN
  - \*14.  $\sim a=b$  12, Simp
  - \*15.  $(y)[(Ha \cdot Hy) \supset a=y]$  12, Com Simp
  - \*16.  $(Ha \cdot Hb) \supset a=b$  2, UI
  - \*17.  $Ha \cdot Hb$  15, UI
  - \*18.  $a=b$  3, 13, Conj
  - \*19.  $a=b \cdot \sim a=b$  16, 17, MP
20.  $(\exists x)[Hx \cdot (y)(Hy \supset x=y)]$  18, 14, Conj